Marches aléatoires renforcées et opérateurs de Schrödinger aléatoires

Xiaolin Zeng

Institut Camille Jordan
UMR 5208 du CNRS

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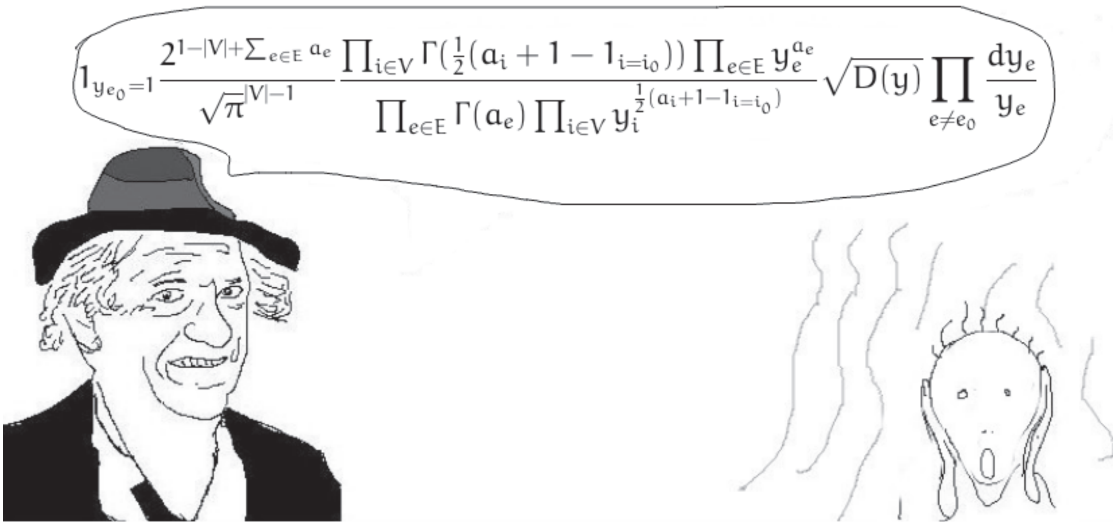

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Thèse de doctorat

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Xiaolin Zeng

devant le jury composé de:

Michel Benaïm	Université de Neuchâtel	Examinateur
Margherita Disertori	Hausdorff Center For Mathematics	Rapporteur
Christophe Garban	Université Lyon 1	Examinateur
Gady Kozma	Weizmann Institute of Science	Rapporteur
Christophe Sabot	Université Lyon 1	Directeur de thèse
Bruno Schapira	Aix-Marseille Université	Examinateur
Zhan Shi	Université Pierre et Marie Curie	Examinateur

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[1]Les clans sont choisis peu (par) pertinents (hasard) et bien évidement j'ai oublié qqn.

J'ai une question pour ceux qui ne connaissent pas Hilbert :
dans ce cas, Que faites-vous dans son espace ?
– blagues mathématiques

Résumé

Cette thèse s'intéresse à deux modèles de processus auto intéagissant étroitement reliés : le processus de sauts renforcé par sites (VRJP) et la marche aléatoire renforcée par arêtes (ERRW). Nous étudions aussi les liens entre ces processus et un opérateur de Schrödinger aléatoire.

Dans le chapitre 3, nous montrons que le VRJP est le seul processus satisfaisant la propriété d'échangeabilité partielle et tel que la probabilité de transition ne dépende que du temps local des voisins, sous quelques conditions techniques.

Le chapitre 4 donne la transition de phase entre vitesse positive et vitesse nulle pour un VRJP transitoire sur un arbre de Galton Watson, utilisant le fait que sur un arbre, le VRJP est une marche aléatoire en milieu aléatoire.

Dans le chapitre 5, une nouvelle famille exponentielle de loi est introduite et ses liens avec le VRJP sont étudiés. En particulier, nous donnons une preuve de la formule de Coppersmith et Diaconis, n'utilisant que des calculs élémentaires.

Finalement, dans le chapitre 6 nous étudions la représentation du VRJP comme mélange de processus de Markov sur les graphes infinis. Nous représentons le VRJP à l'aide de la fonction de Green et d'une fonction propre généralisée d'un opérateur de Schrödinger aléatoire associé au VRJP. En conséquence, nous obtenons un principe d'invariance pour le VRJP quand le renforcement est suffisamment faible, ainsi que la récurrence du ERRW sur $\mathbb{Z}^2$ pour toute valeurs initiales des paramètres.

Abstract

This thesis is dedicated to the study of two closely related self-interacting processes: the vertex reinforced jump process (VRJP) and the edge reinforced random walk (ERRW). We also study the relations between these processes and a random Schrödinger operator.

In Chapter 3, we prove that the VRJP is the only partially exchangeable process whose transition probability depends only on neighbor local times, under some technical conditions.

Chapter 4 gives the phase transition between positive speed and null speed of a transient VRJP on a Galton Watson tree, using a representation of random walk in independent random environment.

In Chapter 5, we introduce a new exponential family of probability distributions generalizing the Inverse Gaussian distribution, and we show some of its relations to the VRJP. In particular, we give an elementary proof of the formula of Coppersmith and Diaconis.

Finally, we show in Chapter 6 that the VRJP on infinite graph is a mixture of Markov jump processes, by constructing the random environment using the Green function and a generalized eigenfunction related to a random Schrödinger operator associated with the VRJP. As a consequence, we obtain a central limit theorem when the reinforcement is weak enough, and also the recurrence of ERRW on $\mathbb{Z}^2$ for any initial constant weights.

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CHAPITRE 1

INTRODUCTION

La marche aléatoire renforcée par arête (ERRW) et le processus de sauts renforcé par site (VRJP) sont deux processus en auto-interaction, qui sont les objets principaux d'étude de cette thèse. Dans cette introduction, nous présentons le contexte dans lequel ils s'insèrent, leurs liens avec des sujets variés, tels que la statistique bayésienne, la diffusion quantique et les sigma modèles hyperboliques. En particulier, nous présentons un résultat récent, qui dit que la marche renforcée par arête est un mélange de processus de sauts renforcé par site, et ce dernier est un mélange de processus Markoviens avec sa loi de mélange explicite. Nous expliquons via un exemple simple comment cette loi est reliée à un sigma-modèle introduit par les physiciens. Finalement nous donnons quelques applications du processus avec renforcement.

Nous énonçons les résultats obtenus durant la thèse dans le chapitre 2, pour intéresser le plus large panel de lecteurs possible, il sera écrit en anglais. Nous expliquons un peu l'idée de la preuve après chaque énoncé. Les chapitres 3–6 représentent les résultats et leurs preuves détaillées, les notations dans ces chapitres peuvent varier les unes des autres, mais elles sont cohérentes dans le chapitre où elles se trouvent.

1.1 Processus avec renforcement

La notion de renforcement s'inspire de certains phénomènes naturels collectifs dans les quels des comportements individuels élémentaires d'apprentissage conduisent à des comportements collectifs structurés. L'exemple classique est celui du comportement d'une colonie de fourmis.

Une colonie de fourmis cherche des sources de nourriture, elles aventurent aux alentours de leur nid de façon aléatoire. Une parmi elles trouve une source de nourriture, elle rend au nid plus au moins directement, en laissant sur son chemin une piste de phéromones attractives ; cela permet aux autres de suivre de façon plus ou moins directe, cette piste.

Grâce à des aléatoires (les 'plus ou moins'), la colonie découvre plusieurs chemins reliant la source et leur nid, les fourmis qui y parcourent ainsi continuent à laisser des pistes de phéromones (à chaque fois un chemin est parcouru, la quantité de phéromones qu'il contient augmente, on dit que le chemin, est renforcé).

Au cours du temps, le chemin le plus court recevra le plus de renforcement, car pour une durée de temps déterminée, il est parcouru le plus souvent. Ainsi, il devient de plus en plus attractif, et les autres chemins vont disparaître petit à petit. A terme, la colonie a donc trouvé le chemin optimal.

Bien que l'intelligence d'une fourmi soit très limitée, mais la colonie, en tant qu'une collection d'individus interagissant suivant des règles simples de renforcement, est efficace pour résoudre ce problème d'optimisation de chemins. Ceci est déjà remarquable, il montre qu'une accumulation du renforcement

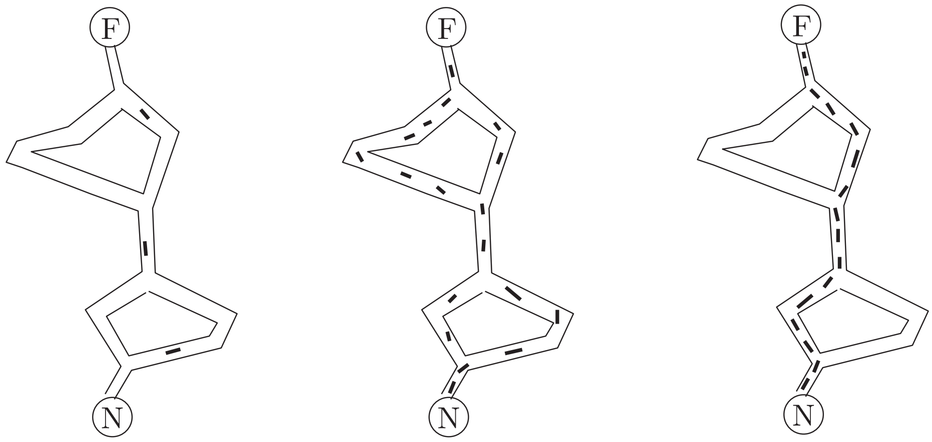

Figure 1.1 : La découverte du chemin optimal

simple peut entraîner des phénomènes non triviaux, comme beaucoup d'autres applications du renforcement (ou plutôt de processus aléatoires renforcés) dans l'activité de l'être humaine que nous présenterons plus tard.

Les modèles de processus avec renforcement les plus populaires sont par exemple : les urnes de Polya, la marche renforcée par arête, la marche renforcée par site, l'attachement préférentiel. Dans cette thèse nous étudions des phénomènes de renforcement à un individu, de façon plus précise nous nous intéressons aux marches aléatoires renforcées qui sont des marches aléatoires qui ont tendance à choisir préférentiellement les directions déjà visitées. L'objet principal d'étude de cette thèse est la marche renforcée linéairement par arête.

Exemple d'illustration : la marche renforcée par arête.

Soit $\{\omega_k, k \geq 0\}$ une suite croissante de réels positifs. La marche renforcée sur $\mathbb{Z}^d$ avec la suite de poids d'arête $\{\omega_k\}$ est le modèle suivant : à chaque temps $n \in \mathbb{N}$, le poids d'une arête est ω_k si cette arête est traversée k fois avant, le processus saute à un voisin de sa position actuelle avec probabilité proportionelle aux poids actuels des arêtes.

1.2 Pourquoi renforcement linéaire ?

Quand le poids est une fonction linéaire du temps local, nous parlons de renforcement linéaire, par exemple, dans le cas de marche renforcée par arête, si les poids $\{\omega_k\}$ s'écrivent

$$\omega_k = a + bk, \text{avec } a, b \in \mathbb{R}^+ \text{ constants.}$$

Le modèle est appelé marche renforcée de façon linéaire par arête (ERRW). Dans cette thèse, nous nous intéressons de façon exclusive au renforcement linéaire, pour essentiellement deux raisons, expliquées dans la suite.

1.2.1 Linéaire, c'est critique

Considérons le cas de marche renforcée par arête sur $\mathbb{Z}^d$, soit $\{\omega_k\}$ le poids des arêtes, nous nous restreignons au cas où

$$\omega_k = 1 + k^p, \; p > 0.$$

D'après Sellke [48], nous avons

Theorem 1.2.1. *La marche renforcée par arête sur $\mathbb{Z}^d$ tel que les poids s'écrivent $\omega_k = 1 + k^p$, $p > 0$, admet les comportements suivants :*

(1) Si $p < 1$, alors la marche visite p.s. une infinité de sites.

(2) Si $p > 1$, la marche visite p.s. un nombre fini de sites et fini par être coincée en une arête (aléatoire).

On voit que le cas linéaire $p = 1$ peut être vu comme critique. En fait, la première fois la marche renforcée est introduite est dans le cas linéaire, en 1986, Coppersmith et Diaconis se sont posé la question suivante : soit $a > 0$, quel est le comportement asymptotique (récurrence/transience) de la marche renforcée par arête sur $\mathbb{Z}^d$ avec $\omega_k = a + k$?

En dimension un, cette question a été répondue par Davis, nous contribuons dans cette thèse à apporter une réponse en dimension deux.

1.2.2 Echangeabilité partielle

Il s'avère que, le cas du renforcement linéaire possède une symétrie remarquable au niveau de sa distribution : l'échangeabilité partielle. La probabilité que la marche renforcée linéaire suive un chemin ne dépend que du nombre de fois où les arêtes sont traversées, mais pas de l'ordre dans laquelle elles sont traversées.

De façon plus précise, soit σ, τ deux chemins partants de même point, on dit que σ et τ sont équivalents si pour toute arête orientée e, le nombre de fois que e est traversée par σ est égal à celui de τ. Alors, une marche aléatoire est dite partiellement échangeable si et seulement si n'importe quels deux chemins équivalents ont la même probabilité.

Par exemple, les trois chemins suivants ont la même probabilité :

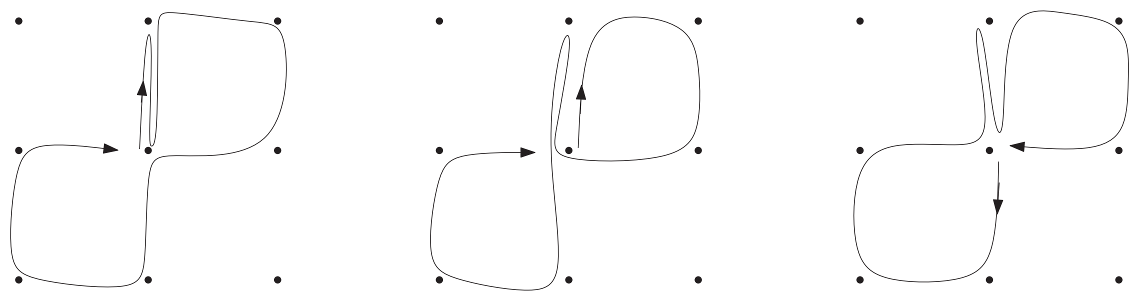

FIGURE 1.2 : Trois chemins équivalents.

On verra plus tard que si nous avons un processus partiellement échangeable et que, ce processus est récurrent, alors, il est, en fait, une marche aléatoire en milieu aléatoire (MAMA). De plus, il est démontré par Rolles [42] que, sous quelques conditions techniques, le ERRW est le seul processus vérifiant cette propriété. Nous montrons dans le chapitre 3 une contrepartie de ce résultat, pour les processus renforcés en temps continu.

1.3 Trois modèles de processus à renforcement linéaire

Nous présentons trois modèles probabilistes avec renforcement linéaire, étroitement liés, le dernier sera notre objet principal d'étude.

1.3.1 Modèle d'urne

Urne de Pólya

On considère une urne contenant a boules rouges et b boules bleues. On tire une boule de l'urne et on la remet avec une autre boule de la même couleur ; puis on recommence et ainsi suite. On appelle ce processus l'urne de Pólya.

De façon mathématique, l'urne de Pólya avec condition initiale (a,b) est un processus discret $(X_n)_{n\geq 1}$ à valeur dans $\{0,1\}$ (donc 0 représente rouge et 1 représente bleue) tel que,

$$\begin{cases} \mathbb{P}(X_1 = 0) = \frac{a}{a+b} \\ \mathbb{P}(X_{n+1} = 0) = \frac{R_n}{R_n+B_n} \quad n \geq 1 \end{cases}$$

où $R_n = a + \sum_{k=1}^n \mathbb{1}_{X_k=0}$ (respectivement $B_n = a + b + n - R_n$) compte le nombre de boules rouges (respectivement bleues) au temps n.

Ce modèle a été introduit par Eggenberger, F. et G. Pólya (1923) [26] pour modéliser la propagation des maladies contagieuses, sa première propriété est la suivante.

Theorem 1.3.1. *La suite de variables aléatoires* $\frac{R_n}{R_n+B_n}$ *converge presque sûrement vers une variable aléatoire* p*, de loi* $Beta(a,b)$*, i.e.* p *est de densité*

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}\mathbb{1}_{[0,1]}(x)dx.$$

En plus, la distribution de ce processus admet une propriété sympathique : si on considère les n premiers tirages $(X_1,\dots,X_n)$, à valeurs dans $\{0,1\}^n$, sa loi ne dépende pas de l'ordre d'apparition des 0 et 1. De façon plus précise, si on note $\sigma = (\sigma_1,\dots,\sigma_n) \in \{0,1\}^n$ et $N_\sigma(0)$ le nombre de 0 dans σ, on a

Proposition 1.3.1. *Pour tout* $n \geq 1$*, si* $\sigma, \tau \in \{0,1\}^n$ *sont tels que* $N_\sigma(0) = N_\tau(0)$*, alors*

$$\mathbb{P}((X_1,\dots,X_n) = \sigma) = \mathbb{P}((X_1,\dots,X_n) = \tau).$$

Les processus vérifiant cette propriété sont dits échangeables, par le théorème de De Finetti 1.3.6, on peut conclure que l'urne de Pólya est un mélange de suite de variables aléatoires i.i.d.

Theorem 1.3.2. *Soit* $(Y_n)_{n\geq 1}$ *un processus défini de la façon suivante : on tire* p *suivant la loi* $Beta(a,b)$*, conditionnement à la valeur de* p*,* $(Y_n)_{n\geq 1}$ *est une suite i.i.d. de variables aléatoires de loi Bernoulli* $\mathrm{B}(1,p)$*. On a égalité en loi entre* (Y_n) *et l'urne de Pólya* (X_n) *partant de condition initiale* (a,b)*.*

Urne de Dirichlet

On n'est pas obligé de se limiter à deux couleurs, le schéma précédent se généralise à m couleurs sans problème. On définit, l'urne de Dirichlet avec condition initiale $(a_1,\dots,a_m)$, le processus $(X_n)_{n\geq 1}$ à valeur dans $\{1,\dots,m\}$ par

$$\begin{cases} \mathbb{P}(X_1 = k) = \frac{a_k}{a_1+\cdots+a_m} \\ \mathbb{P}(X_{n+1} = k) = \frac{N_n(k)+a_k}{\sum_{j=1}^m (N_n(j)+a_j)} \quad n \geq 1 \end{cases}$$

pour tout $k = 1,\dots,m$, où $N_n(k) = \sum_{l=1}^n \mathbb{1}_{X_l=k}$.

On appelle l'intégrale de Dirichlet l'identité suivante :

$$I(a_1,\dots,a_m) = \int_\Sigma (\prod_{k=1}^m x_k^{a_k-1}) dx_1 \cdots dx_{m-1} = \frac{\prod_{k=1}^m \Gamma(a_k)}{\Gamma(\sum_{k=1}^m a_k)}$$

où $\Sigma = \{x_i > 0, \forall i, \sum_{i=1}^m x_i = 1\}$. On dit que $p = (p_1, \dots, p_m)$ suit une loi de Dirichlet de paramètres $(a_1, \dots, a_m)$ et on note $p \sim \text{Dirichlet}(a_1, \dots, a_m)$ si p est de densité sur Σ

$$\frac{1}{I(a_1, \dots, a_m)} \prod_{k=1}^m x_k^{a_k - 1}.$$

Remarquons que la loi bêta peut être considérée comme un cas particulier de loi de Dirichlet, i.e. si $p \sim \text{Beta}(a, b)$ et $(p_1, p_2) \sim \text{Dirichlet}(a, b)$, alors $(p, 1-p) \overset{\text{loi}}{=} (p_1, p_2)$. Il n'est donc pas surprenant que l'on ait les résultats suivants.

Theorem 1.3.3. *Soit $(X_n)_{n\geq 1}$ l'urne de Dirichlet avec condition initiale $(a_1, \dots, a_m)$, on a*

$$\frac{1}{\sum_{k=1}^m N_n(k)} (N_n(1), \dots, N_n(m)) \xrightarrow[n\to\infty]{p.s.} p$$

où $p = (p_1, \dots, p_m) \sim \text{Dirichlet}(a_1, \dots, a_m)$.

Soit $\sigma = (\sigma_1, \dots, \sigma_n) \in \{1, \dots, m\}^n$ et pour tout $1 \leq k \leq m$, on note $N_\sigma(k)$ le nombre de k dans σ, on a

Proposition 1.3.2. *Pour tout $n \geq 1$, si $\sigma, \tau \in \{1, \dots, m\}^n$ sont tels que $N_\sigma(k) = N_\tau(k)$, $\forall 1 \leq k \leq m$, alors*

$$\mathbb{P}((X_1, \dots, X_n) = \sigma) = \mathbb{P}((X_1, \dots, X_n) = \tau).$$

Theorem 1.3.4. *Soit $(Y_n)_{n\geq 1}$ le processus défini de façon suivante : on tire p suivante la loi Dirichlet$(a_1, \dots, a_m)$, conditionnement à p, $(Y_n)_{n\geq 1}$ est i.i.d. et que $\forall 1 \leq k \leq m$, $\mathbb{P}(Y_1 = k) = p_k$. Alors l'urne de Dirichlet (X_n) avec condition initiale $(a_1, \dots, a_m)$ et (Y_n) ont la même loi.*

Tous les théorèmes précédents peuvent être démontrés par le calcul suivant :

Démonstration. Soit p une variable aléatoire de loi Dirichlet$(a_1, \dots, a_m)$, soit (X_n) l'urne de Dirichlet avec condition initiale $(a_1, \dots, a_m)$, soit $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{1, \dots, m\}^n$ une configuration du résultat de tirage. Si on note $(a, k) = a(a+1)(a+2)\cdots(a+k-1)$ et $N_\varepsilon(k) = \text{Card}\{j;\ \varepsilon_j = k, 1 \leq j \leq n\}$, on a

$$\mathbb{P}((X_1, \dots, X_n) = \varepsilon) = \frac{\prod_{k=1}^m (a_k, N_\varepsilon(k))}{(\sum_{k=1}^m a_k, n)}.$$

D'autre part, soit (Y_n) est le processus tel que, conditionné à la valeur de p, $P(Y_n = k) = p_k$ pour tout n. On a

$$\begin{aligned}
\mathbb{P}((Y_1, \dots, Y_n) = \varepsilon) &= \int_\Sigma \prod_{k=1}^m x_k^{N_\varepsilon(k)} \frac{1}{I(a_1, \dots, a_m)} \prod_{k=1}^m x_k^{a_k - 1} dx \\
&= \frac{\Gamma(\sum_{k=1}^m a_k)}{\Gamma(n + \sum_{k=1}^m a_k)} \frac{\prod_{k=1}^m \Gamma(a_k + N_\varepsilon(k))}{\prod_{k=1}^m \Gamma(a_k)} \\
&= \frac{\prod_{k=1}^m (a_k, N_\varepsilon(k))}{(\sum_{k=1}^m a_k, n)}.
\end{aligned}$$

Donc (X) et (Y) ont la même loi et la loi ne dépend que de $\{N_\sigma(k), 1 \leq k \leq m\}$. □

1.3.2 La marche renforcé par arête

On a vu que les urnes avec renforcement linéaire satisfaisant des propriétés sympathiques, une question naturelle à se poser est la suivante : Existe-t-il une contrepartie de l'urne de Dirichelet dans le cas de marche aléatoire ? La réponse est positive et le modèle qui le généralise, appelé marche aléatoire renforcée par arête, admet également des bonnes propriétés ; et de plus, il mène à des problèmes de recherche intéressants et difficiles.

Soit $\mathcal{G}$ un graphe localement fini, non orienté, sans cycle, e.g. des graphes finis, le réseau $\mathbb{Z}^d$ etc. On note V l'ensemble des sites et E l'ensemble des arêtes ; on note $\vec{E}$ pour les arêtes orientées. Chaque arête est étiquetée par un nombre positif a_e, appelé le poids de l'arête. Si $e = \{i, j\} \in E$, on dit que i, j sont voisins et on note $i \sim j$. Parfois il est plus commode d'écrire $a_{i,j}$ pour a_e. On définit le poids d'un site i par $a_i = \sum_{j\sim i} a_{i,j}$.

Une marche aléatoire renforcée par arête partant de i_0 sur $\mathcal{G}$ avec poids initial (a) est un processus $(Z_n)_{n\geq 0}$ à valeurs dans V, tel que, si on note $\mathbb{P}_{i_0}^{(a)}$ sa loi, on a

$$\begin{cases} Z_0 = i_0 \ \mathbb{P}_{i_0}^{(a)}\text{-p.s.} \\ \mathbb{P}_{i_0}^{(a)}(Z_{n+1} = j|\mathcal{F}_n) = \mathbb{1}_{j\sim Z_n} \frac{a_n(\{Z_n,j\})}{\sum_{k\sim Z_n} a_n(\{Z_n,k\})} & n \geq 0 \end{cases}$$

où $\mathcal{F}_n = \sigma(Z_0, \dots, Z_n)$ et pour tout $e \in E,\ n \in \mathbb{N} \cup \{\infty\}$

$$a_n(e) = a_e + \sum_{k=1}^{n} \mathbb{1}_{\{X_{k-1},X_k\}=e}.$$

On introduit une notion de symétrie distributionnelle importante pour les marches renforcées : l'échangeabilité partielle. On dit qu'une suite de sites $\sigma = (i_0, i_1, \dots, i_n)$ dans V est un chemin admissible dans $\mathcal{G}$ si les sites consécutifs sont voisins. Soit σ un chemin admissible, on note, pour $i \in V$, $e = \{i, j\} \in E$, $\vec{e} = (i, j) \in \vec{E}$,

$$\begin{aligned} N_\sigma(i) &= \text{card}(\{k;\ i_k = i,\ k = 0, \dots, n\}) \\ N_\sigma(\vec{e}) &= N_\sigma((i, j)) = \text{card}(\{k;\ i_k = i, i_{k+1} = j,\ k = 0, \dots, n-1\}) \\ N_\sigma(e) &= N_\sigma(\{i, j\}) = \text{card}(\{k;\ \{i_k, i_{k+1}\} = \{i, j\},\ k = 0, \dots, n-1\}) \end{aligned}$$

qui représente respectivement le nombre de visites d'un site i, le nombre de fois la marche a traversé l'arête orientée $\vec{e}$ ou l'arête non orientée e. Par convention, on note $N_n(\cdot)$ pour $N_\sigma(\cdot)$ quand $\sigma = (Z_0, \dots, Z_n)$. En particulier, $a_n(e) = a_e + N_n(e)$.

Deux chemins admissibles σ, τ sont dits équivalents si $N_\sigma(\vec{e}) = N_\tau(\vec{e})$ pour tout $\vec{e} \in \vec{E}$ et on note $\sigma \sim \tau$.

Remark 1.3.1. *Deux chemins équivalents ont toujours la même longueur et terminent par le même site.*

Definition 1.3.1. *Une marche aléatoire au plus proche voisin sur $\mathcal{G}$ est partiellement échangeable si n'importe quels deux chemins équivalents ont la même probabilité ; i.e., la probabilité $\mathbb{P}(X \sim \sigma)$ ne dépend que de $\{N_\sigma(\vec{e}), \vec{e} \in \vec{E}\}$. Si, entre autres, cette probabilité ne dépend que de $\{N_\sigma(e), e \in E\}$, alors on dit que la marche est partiellement échangeable au sens réversible.*

Proposition 1.3.3. *La marche renforcée par arête est partiellement échangeable (au sens réversible).*

Démonstration. Soit Z_n la ERRW partant de i_0 avec poids initiaux (a), soient σ, τ deux chemins admissibles équivalents partant de i_0, on a

$$\mathbb{P}_{i_0}^{(a)}(\sigma) = \frac{\prod_{e\in E}(a_e(a_e + 1) \cdots (a_e + N_\sigma(e) - 1))}{\prod_i (a_i(a_i + 1) \cdots (a_i + N_\sigma(i) - 1 - \mathbb{1}_{i=i_0}))}$$

qui ne dépend que de $N_\sigma(e), e \in E$, donc $\mathbb{P}_{i_0}^{(a)}(\sigma) = \mathbb{P}_{i_0}^{(a)}(\tau)$. $\square$

1.3.3 Théorème de De Finetti et formule magique

Avant d'aller plus loin, nous allons démontrer un résultat très joli et profond dans cette section, qui sera aussi très important pour la suite : Le théorème de De Finetti pour les chaînes de Markov. Étant donné un graphe $\mathcal{G} = (V, E)$, à chaque arête $e \in E$ on associe un réel positif y_e, appelé sa conductance. La chaîne de Markov à conductance (y) est définie par sa probabilité de transition suivante :

$$p(i, j) = \mathbb{1}_{i\sim j} \frac{y_{i,j}}{y_i}$$

où $y_i = \sum_{j:j\sim i} y_{i,j}$. Si de plus les conductances (y) sont aléatoires, on peut intégrer la loi de la chaîne par rapport à la loi de (y) pour obtenir un mélange de chaînes de Markov, ou autrement dit une marche aléatoire en milieu aléatoire (MAMA).

Formellement, soit (y) distribué suivant certaine loi $\mu(dy)$, soit $P_{i_0}^{(y)}(\cdot)$ la loi de la chaîne de Markov réversible à conductance (y) partant de i_0 ; i.e. avec probabilité de transition $p(i, j)$. La marche aléatoire de probabilité

$$\mathbb{P}_{i_0}(\cdot) = \int P_{i_0}^{(y)}(\cdot) d\mu(y)$$

est appelée la marche aléatoire partant de i_0 en environnent aléatoire μ.

Voici le résultat fameux qui relie échangeabilité partielle et MAMA :

Theorem 1.3.5 (Diaconis & Freedman[20]). *Soit (Z_n) une marche aléatoire récurrente (i.e. la marche revient au point de départ Z_0 infiniment souvent p.s.), Z_n est une MAMA si et seulement si elle est partiellement échangeable, de plus, la mesure de mélange est unique.*

Avant de donner la preuve, introduisons brièvement le théorème de De Finetti pour les suites échangeables. Une suite infinie dénombrable $(\xi_i)_{i\in I}$ d'éléments aléatoires dans un espace de Borel S est dit échangeable si

$$(\xi_{k_1}, \dots, \xi_{k_m}) \stackrel{d}{=} (\xi_1, \dots, \xi_m)$$

pour toute famille $k_1, \dots, k_m$ d'éléments distincts dans l'ensemble des indices I de ξ.

Theorem 1.3.6 (De Finetti). *Une suite infinie d'éléments aléatoires dans un espace de Borel S est échangeable si et seulement si elle est conditionnellement i.i.d. (i.e. il existe une sous tribu $\mathcal{F}$ telle que la suite est i.i.d. conditionnée à $\mathcal{F}$).*

Démonstration. [1] Soit $(\xi_n)_{n\in\mathbb{N}}$ une suite échangeable. Pour toute injection $\gamma : \mathbb{N}^* \to \mathbb{N}^*$, en considérant les marginales fini dimensionnelles, on a

$$(\xi_i)_i \stackrel{d}{=} (\xi_{\gamma(i)})_i.$$

En particulier, on peut construire une famille de variables aléatoires $(\xi_i)_{i\in\mathbb{Z}}$ échangeables tells que $(\xi_i)_{i\in\mathbb{N}}$ est distribués comme notre suite de départ. Soit $Z = (\xi_i)_{i\in\mathbb{Z}\setminus\mathbb{N}^*}$, on a, pour toute permutation (finie) σ, grâce à l'échangeabilité

$$(Z, \xi_1, \xi_2, \cdots) \stackrel{d}{=} (Z, \xi_{\sigma(1)}, \xi_{\sigma(2)}, \cdots).$$

De plus, les variables aléatoires ξ_i sont indépendants conditionnées à la valeurs de Z. En fait, pour toute fonction de teste f mesurable et bornée,

$$\mathbb{E}(f(\xi_k)|Z) = \mathbb{E}(\mathbb{E}(f(\xi_k)|Z, \xi_1, \dots, \xi_{k-1})|Z).$$

[1] Cette preuve est survolée d'Austin

En particulier, $\|\mathbb{E}(f(\xi_k)|Z)\|_2 \leq \|\mathbb{E}(f(\xi_k)|Z, \xi_1, \ldots, \xi_{k-1})\|_2$, puis par l'échangeabilité, ceci est en fait une égalité, ainsi (vu $\mathbb{E}(f(\xi_k)|Z)$ comme une projection dans l'espace d'Hilbert)

$$\mathbb{E}(f(\xi_k)|Z) = \mathbb{E}(f(\xi_k)|Z, \xi_1, \ldots, \xi_{k-1}) \text{ p.s.}$$

En récapitulatif,

$$\begin{cases} (Z, \xi_i) \overset{d}{=} (Z, \xi_j), \ \forall i, j \\ \xi_1, \xi_2, \cdots \text{ sont indépendants conditionné à } Z. \end{cases}$$

Ce qui veut dire que les (ξ_i) sont conditionnement i.i.d. □

Démonstration du théorème de De Finetti pour les chaînes de Markov. On se contente du cas $Z_n \in \mathbb{N}$, supposons que $\mathbb{P}(Z_0 = 1) = 1$, on définit les 1-blocs de Z_n comme les chaînes finies des états commençant par 1 qui ne contient plus des 1's, par l'échangeabilité partielle, on a

les 1-blocs sont échangeable.

Soient $Y_0, Y_1, \cdots$ ces 1-blocs, on note $\mathcal{F}_\infty$ la tribu asymptotique de Y, par le théorème de De Finetti, conditionnement à $\omega \in \mathcal{F}_\infty$, Y_i sont i.i.d. Soit P^ω cette probabilité conditionnelle, nous montrerons que, pour presque tout ω, $\sigma \sim \tau$ entraîne

$$P^\omega(A_\sigma) = P^\omega(A_\tau) \tag{1.1}$$

où $A_\sigma = \{Z_0 = i_0, Z_1 = i_1, \cdots, Z_k = i_k\}$ avec $\sigma = (i_0, i_1, \ldots, i_k)$.

Il suffit de démontrer que, pour n suffisamment grand, pour tout m, pour toute valeurs possibles des 1-blocs $\beta_0, \cdots, \beta_m$,

$$\mathbb{P}(A_\sigma | Y_n = \beta_0, \cdots, Y_{n+m} = \beta_m) = \mathbb{P}(A_\tau | Y_n = \beta_0, \cdots, Y_{n+m} = \beta_m).$$

Soit $n > N_\sigma(1)$, on considère l'ensemble des chaînes ψ telles que $A_{\sigma\psi}$ est un chemin avec $Y_n = \beta_0, \cdots, Y_{n+m} = \beta_m$. Pour tout telles ψ, $\sigma\psi \sim \tau\psi$, donc, $\mathbb{P}(A_{\sigma\psi}) = \mathbb{P}(A_{\tau\psi})$, l'égalité exigée est obtenu en sommant sur les ψ.

Il reste à montrer la propriété de Markov conditionnelle, i.e. si σ, σ' sont deux chaînes commençant par 1 et termine par certain i, non nécessairement équivalent, on a, pour presque tout $\omega \in \mathcal{F}_\infty$,

$$P^\omega(A_{\sigma j} | A_\sigma) = P^\omega(A_{\sigma' j} | A_{\sigma'}). \tag{1.2}$$

Remarquons que conditionnellement à ω, les 1-blocs sont i.i.d., si on prend α,β deux chaînes ne contenant pas 1, on a $P^\omega(A_{1\alpha 1\beta}) = P^\omega(A_{1\beta 1\alpha})$. En considérant toutes les chaînes finies ψ ne contenant pas 1, comme $\sigma\psi\sigma' j \sim \sigma'\psi\sigma j$, par (1.1) et conditionnellement i.i.d.

$$P^\omega(A_{\sigma\psi 1}) P^\omega(A_{\sigma' j}) = P^\omega(A_{\sigma\psi\sigma' j}) = P^\omega(A_{\sigma'\psi\sigma j}) = P^\omega(A_{\sigma'\psi 1} P^\omega_{\sigma j}).$$

En sommant sur toutes les possibilités de ψ, on a

$$P^\omega(A_\sigma) P^\omega(A_{\sigma' j}) = P^\omega(A_{\sigma'}) P^\omega(A_{\sigma j}).$$

□

Proposition 1.3.4. *Soit $\mathcal{G}$ un graphe fini, le ERRW sur $\mathcal{G}$ traverse p.s. toute arête infiniment souvent dans les deux directions.*

Démonstration. Soit $\vec{e} = (i, j)$, notons X_n le ERRW avec poids initiaux $(a$ partant de $\rho \in V$. Soit τ_k le $k^{\text{ème}}$ visite au site i. Soit $A(j, i)$ l'événement

$$\{j \text{ est visité infiniment souvent}, (j, i) \text{ est traversé au plus un nombre fini de fois}\}$$

On a

$$\mathbb{P}_\rho^{(a)}(A(j,i)) = \lim_{k_0\to\infty} \lim_{K\to\infty} \mathbb{P}_\rho^{(a)}(\cap_{k_0\le k\le K}\{\tau_k < \infty, X_{\tau_k+1} \neq i\}).$$

Comme

$$\mathbb{P}_\rho^{(a)}(X_{\tau_k+1} \neq i | \mathcal{F}_{\tau_k}) = 1 - \frac{N_{\tau_k}((j,i))}{N_{\tau_k}(j)} \le 1 - \frac{a_e}{a_j + 2k} \le e^{-\frac{a_e}{a_j+2k}},$$

par induction

$$\mathbb{P}_\rho^{(a)}(\cap_{k_0\le k\le K}\{\tau_k < \infty, X_{\tau_k+1} \neq i\}) \le \prod_{k_0\le k\le K} e^{-\frac{a_e}{a_j+2k}}.$$

Si on fait d'abord $K \to \infty$, puis $k_0 \to \infty$, on voit que $\mathbb{P}_\rho^{(a)}(A(j,i)) = 0$. Donc, si j est visité infiniment souvent, alors toutes les arêtes adjacentes sont visitées infiniment souvent, il en est de même pour tous ses voisins. Maintenant il suffit de remarquer que, sur un graphe fini, au moins un site est visité infiniment souvent par la marche aléatoire. □

En utilisant Théorème 1.3.5 et Proposition 1.3.4, on sait que, sur un graphe fini, tout ERRW est une MAMA, en particulier, il existe une unique mesure de mélange. En fait, cette mesure de mélange est connue de façon explicite sous le nom de mesure de Coppersmith-Diaconis ou 'formule magique'.

Theorem 1.3.7 (Mesure de Coppersmith-Diaconis [16],[31]). *Soit Z_n un ERRW sur un graphe fini $\mathcal{G} = (V, E)$ partant de i_0 avec poids initiaux a. On note $V = (v_1, \dots, v_{|V|})$ avec $|V|$ le cardinal de V. Soit e_0 une arête fixée contenant i_0, et $\mathcal{H}_{e_0} = \{(y_e)_{e\in E};\ y_{e_0} = 1, \forall e \in E,\ y_e > 0\}$. Si on note*

$$\mathcal{M}_{i_0}^{(a)}(dy) = C(a, i_0) \frac{\sqrt{y_{i_0}} \prod_{e\in E} y_e^{a_e}}{\prod_{i\in V} y_i^{(a_i+1)/2}} \sqrt{D(y)} \prod_{e\neq e_0} \frac{dy_e}{y_e} \tag{1.3}$$

où $D(y)$ est n'importe quel mineur diagonal de la matrice $\begin{pmatrix} -y_{v_1} & y_{v_1,v_2} & \cdots & y_{v_1,v_{|V|}} \\ & \cdots & \cdots & \\ y_{v_{|V|},v_1} & \cdots & \cdots & -y_{v_{|V|}} \end{pmatrix}$ *et*

$$C(a, i_0) = \frac{2^{1-|V|+\sum_{e\in E} a_e}}{\sqrt{\pi}^{|V|-1}} \cdot \frac{\prod_{i\in V} \Gamma(\frac{1}{2}(a_i + 1 - \mathbb{1}_{i=i_0}))}{\prod_{e\in E} \Gamma(a_e)},$$

alors $\mathcal{M}_{i_0}^{(a)}$ est une mesure de probabilité sur $\mathcal{H}_{e_0}$, qui est la mesure de mélange de Z_n, i.e.

$$\mathbb{P}_{i_0}^{(a)}(\cdot) = \int_{\mathcal{H}_{e_0}} P_{i_0}^{(y)}(\cdot) \mathcal{M}_{i_0}^{(a)}(dy)$$

où $P_{i_0}^{(y)}$ est la probabilité pour la chaîne réversible à conductance y partant de i_0.

En particulier, on sait que l'intégrale de (1.3) vaut 1, ce qui n'est, si on oublie les modèles probabilistes derrière, complètement pas une identité triviale. Diaconis avait posé la question suivante : serait il possible de montrer $\int \mathcal{M}_{i_0}^{(a)}(dy) = 1$ par un calcul direct ? Nous donnons une réponse positive à cette question dans le chapitre 5.

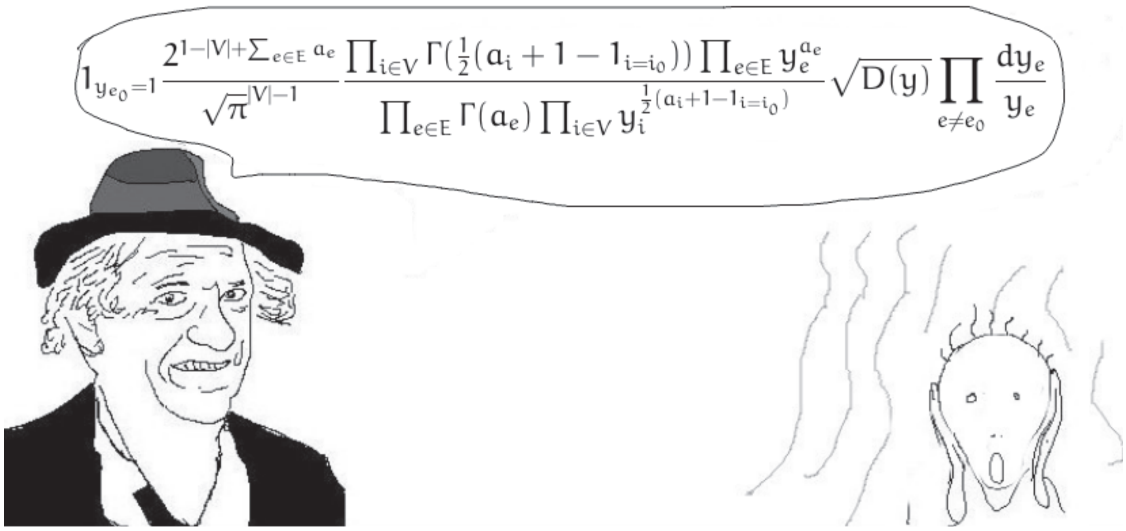

FIGURE 1.3 : La formule magique et le magicien

1.3.4 Le processus de sauts renforcé par site

Passons au modèle principale de nos études, le processus de sauts renforcé par site (VRJP).

Définition et relation avec ERRW

Soit $\mathcal{G} = (V, E)$ un graphe non orienté localement fini, pour chaque arête $e = \{i, j\}$ on associe un réel positive $W_{i,j}$, appelé la conductance de l'arête. Soit Y_t un processus à temps continu à valeur dans V. Le temps local de Y au site i est défini par

$$L_i(t) = 1 + \int_0^t \mathbb{1}_{Y_u=i} du.$$

Remark 1.3.2. *On peut remplacer* 1 *par* ϕ_i *dans la définition du temps local, cela revient à changer la conductance* W.

On dit que Y_t est un VRJP partant de i_0 avec conductance W, si $Y_0 = i_0$ p.s. et, au temps t, Y_t saute vers un voisin j de sa position actuelle à taux

$$W_{Y_t,j} L_j(t)$$

et on note $\mathbb{P}_{i_0}^W$ la mesure de probabilité de Y_t. La relation entre le VRJP et ERRW sur les graphes finis est la suivante.

Theorem 1.3.8 (Sabot et Tarrès [45]). *Le ERRW* (Z_n) *de poids initiaux* a *est égal en loi au processus en temps discrète associé à un VRJP en conductances aléatoires indépendantes* $W_e \sim \text{Gamma}(a_e, 1)$.

Ce théorème, reliant le ERRW et le VRJP, montre que la recherche du comportement de ERRW peut être ramenée à l'étude du VRJP. La preuve de ce théorème s'appuie sur un résultat de Kendall pour les processus de branchement et la construction de Rubin.

Un résultat de Kendall sur le processus de Yule

Le processus de branchement permet une modélisation simple de la reproduction des cellules. Imaginons que les cellules se reproduisent selon les règles suivantes (où $\lambda > 0$ est le taux de reproduction) :

1. Une cellule présente au temps t se divise en deux dans l'intervalle de temps $(t, t+h)$ avec probabilité $\lambda h + o(h)$.

2. Cette probabilité est indépendante de l'âge.

3. Les événements entre différentes cellules sont indépendants.

Soit $\mathcal{Y}_t$ le nombre de cellules au temps t, avec $\mathcal{Y}_0 = a > 0$. On appelle un tel processus le processus de Yule ou le processus de naissance pure avec taux de reproduction λ. Il peut être aussi considéré comme un processus ponctuel sur la droite réelle. Soit $a \in \mathbb{R}_+^*$, posons $\tau_1, \tau_2, \cdots$ des variables aléatoires exponentielles de paramètres $\lambda a, \lambda(a+1), \cdots$. On pose $\mathcal{Y}_0 = a$ et il est facile de vérifier l'égalité suivante :

$$\mathcal{Y}_t = a - 1 + \inf\{k \geq 1 | \tau_1 + \cdots + \tau_k > t\}.$$

Avec cette définition on voit que a n'est pas obligatoirement un entier.

Proposition 1.3.5. *Soit $\mathcal{Y}_t$ le processus de Yule avec condition initiale $\mathcal{Y}_0 = 1$, alors $W_t = \mathcal{Y}_t e^{-\lambda t}$ est une martingale et elle converge vers W, une v.a. exponentielle de paramètre* 1.

Remark 1.3.3. *De façon plus générale, le processus de Yule avec $\mathcal{Y}_0 = a$ satisfait $\mathbb{E}_a(\mathcal{Y}_t) = ae^{\lambda t}$ et la martingale $W_t = \mathcal{Y}_t e^{-\lambda t}$ converge vers W de transformée de Laplace $\frac{1}{(1+\theta)^a}$, donc W est de loi Gamma$(a, 1)$.*

Theorem 1.3.9. *Soit $\mathcal{Y}_t$ le processus de Yule à taux λ, et $\mathcal{Y}_0 = a$. Conditionnellement à $W = \lim_t \mathcal{Y}_t e^{-\lambda t}$ qui suit une loi de Gamma$(a, 1)$, le processus $(\mathcal{Y}_{f(t)}, t \geq 0)$ est un processus ponctuel de Poisson de paramètre* 1*, où*

$$f(t) = \frac{1}{\lambda} \log(1 + \frac{t}{W}).$$

Pour être autonome, nous donnons les preuves de ces deux résultats à la fin de cette section.

Rubin's construction

Maintenant on applique le résultat de Kendall. Soit $G = (V, E)$ un graphe fini, soit $a = (a_e, \ e \in E)$ les poids initiaux. On définit le ERRW en temps continu introduit par Rubin, Davis et Sellke, noté $(\tilde{Z}_t, t \in \mathbb{R}^+, \tilde{Z}_0 = i_0)$ par :

1. Sur chaque arête $e \in E$ on définit des processus de Yule indépendant de population initiale a_e.

2. A chaque arête $e \in E$ on associe une alarme, qui tourne si et seulement si $\tilde{Z}_t$ est adjacent à e. Cet alarme sonne à chaque point de son processus de Yule associé.

3. Quand l'alarme sur $e \in E$ sonne, $\tilde{Z}_t$ traverse e immédiatement.

Theorem 1.3.10 (Davis, Sellke)**.** *Soit Z_n le ERRW avec poids initiaux a, partant de $i_0 \in V$, le processus discret associé à $\tilde{Z}_t$ et Z_n ont la même loi.*

Démonstration. Vu que les variables exponentielles sont sans mémoire, si au temps t, $\tilde{Z}_t = i$, alors les alarmes sur les arêtes adjacentes à i sont de loi exponentielle de paramètres respectivement $a_e + N_t(e)$, où $N_t(e)$ est le nombre de fois l'arête e est traversée par le processus $\tilde{Z}$ jusqu'au temps t, ainsi la probabilité de sauter au voisin j est $\frac{a_{i,j}+N_t(i,j)}{\sum_k a_{i,k}+N_t(i,k)}$, qui n'est rien d'autre que la probabilité de transition de Z_n. □

Theorem 1.3.11. *Le ERRW en temps continu $\tilde{Z}_t$, conditionné à W_e, où W_e est la limite des processus de Yule partant de a_e, saute de i vers j au temps t à taux $W_{i,j} e^{T_i(t)+T_j(t)}$, où $T_i(t) = \int_0^t \mathbb{1}_{\tilde{Z}_u = i} du$.*

Démonstration. Soit $f_{W_e}(t) = \log(1+t/W_e)$, Par le résultat de Kendall 1.3.9, conditionné à W_e, le processus ponctuel sur l'arête e au temps t saute à taux $(f_{W_e}^{-1})'(t) = W_e e^t$. Rappelons que l'alarme ne fonctionne que si le processus est adjacent à e, il résulte qu'au temps t du processus ERRW, le processus ponctuel sur l'arête $e = \{i, j\}$ est à son propre temps $T_i(t) + T_j(t)$, donc le taux de saut est $W_{i,j} e^{T_i(t)+T_j(t)}$. □

Changement de temps

Proposition 1.3.6. *Soit Y_t le VRJP de conductance W. Posons $A(t) = \sum_i \log L_i(t)$, $X_t = Y_{A^{-1}(t)}$. Alors X_t saut de i vers j au temps t à taux $W_{i,j}e^{T_i(t)+T_j(t)}$, où $L_i(t) = 1 + \int_0^t \mathbb{1}_{Y_u=i}du$ et $T_i(t) = \int_0^t \mathbb{1}_{X_u=i}du$.*

Démonstration. On écrit

$$\begin{aligned}\mathbb{P}(X_{t+dt} = j|\mathcal{F}_t, X_t = i) &= \mathbb{P}(Y_{A^{-1}(t+dt)} = j|\mathcal{F}_t, X_t = i)\\ &= W_{i,j}L_j(A^{-1}(t))d(A^{-1}(t))\\ &= W_{i,j}L_j(A^{-1}(t))(A^{-1}(t))'dt.\end{aligned}$$

comme le changement de temps est fait site par site, $T_i(A(t)) = \log L_i(t)$, donc,

$$e^{T_i(t)} = L_i(A^{-1}(t)),\ (A^{-1})'(t) = L_i(A^{-1}(t)) = e^{T_i(t)}$$

par conséquent, le taux de saut est $W_{i,j}e^{T_i(t)+T_j(t)}$. □

Pour finir la preuve du Théorème 1.3.8, remarquons que par Théorème 1.3.9, le ERRW en temps continus $\tilde{Z}_t$ peut-être considéré comme un mélange de processus à taux $W_{i,j}e^{T_i(t)+T_j(t)}$, avec les conductances aléatoires $W_{i,j} \sim \text{Gamma}(a_{i,j}, 1)$.

Factorisation de la formule magique

Le VRJP Y_t n'est clairement pas un mélange de processus de Markov, comme il accélère. En fait, l'accélération peut être compensée par un changement du temps approprié, et le processus dans la bonne échelle devient un mélange de processus de Markov. Plus précisément, posons

$$D(t) = \sum_{i\in V}(L_i(t)^2 - 1),$$

le processus $Z_t = Y_{D^{-1}(t)}$ est un processus (en tant qu'une version de Y par changement de temps) tel que, si on note

$$S_i(t) = \int_0^t \mathbb{1}_{Z_u=i}du$$

le temps local de Z du site i, on a la proposition suivante :

Proposition 1.3.7. *Conditionné au passé, Z_t saute à un voisin j à taux*

$$\frac{1}{2}W_{Z_t,j}\sqrt{\frac{S_j(t)+1}{S_{Z_t}(t)+1}}.$$

Démonstration. En fait, on a (pour simplifier, on écrit $\mathbb{P} = \mathbb{P}^W_{i_0}$)

$$\begin{aligned}\mathbb{P}(Z_{t+dt} = j|\mathcal{F}_t) &= \mathbb{P}(Y_{D^{-1}(t+dt)} = j|\mathcal{F}_t)\\ &= W_{Z_t,j}L_j(D^{-1}(t))d(D^{-1}(t))\\ &= W_{Z_t,j}L_j(D^{-1}(t))(D^{-1}(t))'dt.\end{aligned}$$

Calculons $L_j(D^{-1}(t))$ et $(D^{-1}(t))'$, comme

$$D(t) = \sum_{i\in V}(L_i(t)^2 - 1) = \sum_{i\in V} S_i(D(t))$$

et que le changement de temps est fait site par site, on a $S_i(D(t)) = L_i(t)^2 - 1$, donc

$$L_i(D^{-1}(t)) = \sqrt{S_i(t)+1};$$

d'ailleurs, comme $D'(t) = 2L_i(t)\mathbb{1}_{Y_t=i}dt$, on a

$$(D^{-1}(t))' = \frac{1}{D'(D^{-1}(t))} = \frac{\mathbb{1}_{Z_t=i}}{2L_i(D^{-1}(t))},$$

par conséquent

$$\mathbb{P}(Z_{t+dt} = j|\mathcal{F}_t) = \frac{1}{2}W_{Z_t,j}\sqrt{\frac{S_j(t)+1}{S_{Z_t}(t)+1}}.$$

□

Theorem 1.3.12 (Sabot& Tarrès 2012)**.** *Le VRJP changé de temps (Z_t) partant de i_0 avec conductances W, est un mélange de processus de Markov. Plus précisément, si $\{u(i),\ i \in V\}$ est le vecteur aléatoire à densité $Q^W_{i_0}(u)$, qui s'exprime de la façon suivante*

$$\frac{\mathbb{1}_{u(i_0)=0}}{\sqrt{2\pi}^{|V|-1}}\exp\left(-\sum_i u(i) - \frac{1}{2}\sum_{i\sim j} W_{i,j}[e^{u(j)-u(i)} + e^{u(i)-u(j)} - 2]\right)\sqrt{D(W,u)} \tag{1.4}$$

où $D(W,u)$ est n'importe quel mineur diagonal de matrice laplacienne du graphe avec $W_{i,j}e^{u(i)+u(j)}$ comme le poids de l'arête $\{i,j\}$, alors Z_t est égal en loi au processus annelaed de MAMA avec environnement u partant de i_0 à taux $\frac{1}{2}W_{i,j}e^{u(j)-u(i)}$.

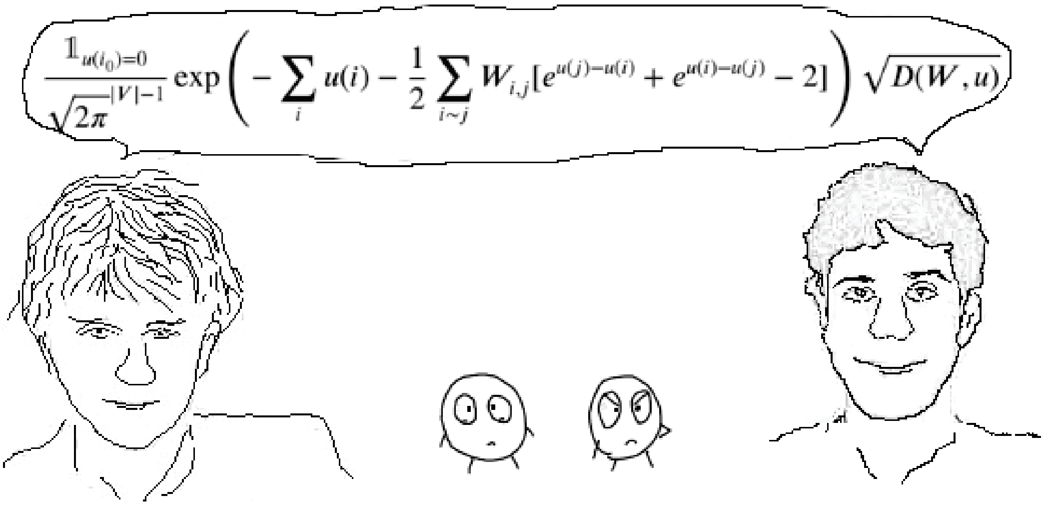

FIGURE 1.4 : Encore une autre formule magique

Nous présentons une proposition qui permet de retrouver la formule magique avec la densité (1.4). Voir la section 5 du [45] pour une preuve.

Proposition 1.3.8. *Soit $W_{i,j}$ de loi Gamma avec paramètre $(a_{i,j}, 1)$, i.e.*

$$W_{i,j} \sim \frac{(W_{i,j})^{a_{i,j}}e^{-W_{i,j}}}{\Gamma(a_{i,j})}\frac{dW_{i,j}}{W_{i,j}},$$

alors l'espérance de $Q^W_{i_0}$ par rapport à W est la formule magique (1.3).

Dans [45], un lien a été établi entre le VRJP et un modèle de théorie quantique des champs relié au modèle d'Anderson [23, 24]. Ce lien a en particulier permis d'établir la récurrence forte de la marche renforcée en toutes dimensions pour des grands renforcements, et la transience en dimension plus grande que 3 pour des faibles renforcements.

Preuves des résultats pour le processus de Yule

Démonstration de la proposition 1.3.5. Montrons d'abord que $\mathbb{E}(\mathcal{Y}_t) = e^{\lambda t}$. Pour $0 < i < j$ et $t > 0$, soit

$$P_{i,j}(t) = \mathbb{P}(\mathcal{Y}_t = j | \mathcal{Y}_0 = i),$$

L'équation de Kolmogorov backward s'écrit

$$\frac{d}{dt} P_{i,j}(t) = i\lambda(P_{i+1,j}(t) - P_{i,j}(t)).$$

Posons $F_a(s,t) = \mathbb{E}(s^{\mathcal{Y}_t} | \mathcal{Y}_0 = a) = \sum_{k=0}^{\infty} P_{a,a+k}(t) s^{a+k}$, pour $a, b > 0$, on a

$$\begin{aligned} F_a(s,t) F_b(s,t) &= \left(\sum_{k=0}^{\infty} P_{a,a+k}(t) s^{a+k} \right) \left(\sum_{k=0}^{\infty} P_{b,b+k}(t) s^{b+k} \right) \\ &= \sum_{k=0}^{\infty} P_{a+b,a+b+k}(t) s^{a+b+k} = F_{a+b}(s,t). \end{aligned}$$

en particulier

$$F_a(s,t) = F_1(s,t)^a.$$

Donc il suffit de considérer le cas $\mathcal{Y}_0 = 1$. Dans la suite on note $F(s,t) = F_1(s,t)$. A nouveau par l'équation backward :

$$\begin{aligned} \frac{\partial}{\partial t} F(s,t) &= \frac{\partial}{\partial t} \sum_{k=0}^{\infty} P_{1,1+k}(t) s^{1+k} \\ &= \sum_{k=0}^{\infty} \frac{\partial}{\partial t} P_{1,1+k}(t) s^{1+k} \\ &= \sum_{k=0}^{\infty} \lambda(P_{2,1+k}(t) - P_{1,1+k}(t)) s^{1+k} \\ &= \lambda(F(s,t)^2 - F(s,t)). \end{aligned}$$

La solution de cette EDP quasi-linéaire du premier ordre est

$$F(s,t) = \frac{1}{e^{\lambda t + c(s)} + 1},$$

en utilisant la condition initiale $F(s,0) = s$ on a

$$F(s,t) = \frac{s e^{-\lambda t}}{1 - (1 - e^{-\lambda t}) s}.$$

En plus, Si on dérive l'EDP par rapport à s et on fait $s \to 1$, on obtient

$$\frac{\partial}{\partial t} \mathbb{E}(\mathcal{Y}_t) = \lambda \mathbb{E}(\mathcal{Y}_t)$$

Avec condition initiale $\mathbb{E}(\mathcal{Y}_0) = 1$ on a $\mathbb{E}(\mathcal{Y}_t) = e^{\lambda t}$. Pour montrer que $W_t = \mathcal{Y}_t e^{-\lambda t}$ est une martingale. Il suffit de remarquer que exactement le même argument montrant $\mathbb{E}(\mathcal{Y}_t) = e^{\lambda t}$ nous donne,

$$\mathbb{E}(\mathcal{Y}_t|\mathcal{F}_s) = \mathcal{Y}_s e^{\lambda(t-s)}.$$

Donc W_t est une martingale, comme W_t est positive, elle converges vers une v.a. positive W. La transformée de Laplace de W_t est (par l'expression de $F(s,t)$)

$$\begin{aligned}\mathbb{E}(e^{-\theta W_t}) &= \mathbb{E}((e^{-\theta e^{-\lambda t}})^{\mathcal{Y}_t})\\ &= \frac{se^{-\lambda t}}{1-(1-e^{-\lambda t})s} \text{ où } s = e^{-\theta e^{-\lambda t}}\\ &\xrightarrow{t\to\infty} \frac{1}{1+\theta}.\end{aligned}$$

Ce qui est aussi la transformée de Laplace d'une v.a. exponentielle de paramètre 1. □

Démonstration du théorème 1.3.9. On se contente du cas $\mathcal{Y}_0 = 1$. Soient $t_1, t_2, \cdots$ les temps de saut de processus $\mathcal{Y}_t$. Comme $f^{-1}(s) = W(e^{\lambda s} - 1)$, il suffit de montrer que, conditionnellement à W,

$$W(e^{\lambda t_j} - e^{\lambda t_{j-1}}),\ j = 1, 2, \cdots$$

sont i.i.d. de distribution $\mathcal{E}xp(1)$ et indépendant de W. Soit $k \geq 1$, $a_0, a_1, \cdots, a_k \geq 0$, il suffit de montrer que

$$E_k = \mathbb{E}(\exp(-W(a_0 + \sum_{j=1}^{k} a_j(e^{\lambda t_j} - e^{\lambda t_{j-1}})))) = \prod_{j=0}^{k} \frac{1}{1+a_j}.$$

Par la propriété branchante, on a

$$W = e^{-\lambda t_k} \sum_{j=1}^{k+1} W_k^{(j)}$$

où $W_k^{(j)}$ sont des limites des copies indépendantes des processus de Yule : $W_k^{(j)} = \lim_t Z_t^{(j)} e^{-\lambda t}$. Il vient

$$\begin{aligned}E_k &= \mathbb{E}(\exp(-W(a_0 + \sum_{j=1}^{k} a_j(e^{\lambda t_j} - e^{\lambda t_{j-1}}))))\\ &= \mathbb{E}(\exp(-\underbrace{e^{-\lambda t_k}(a_0 + \sum_{j=1}^{k} a_j(e^{\lambda t_j} - e^{\lambda t_{j-1}})}_{g_k}\sum_{j=1}^{k+1} W_k^{(j)}))\\ &= \mathbb{E}[\mathbb{E}(\exp(-g_k \sum_{j=1}^{k} W_k^{(j)}|\mathcal{F}_k))], \text{ remarque que } g_k \text{ est } \mathcal{F}_k \text{ mesurable}\\ &= \mathbb{E}(\frac{1}{(1+g_k)^{k+1}}), \text{ comme conditionné à } \mathcal{F}_k,\ W_k^{(j)} \text{ sont i.i.d. de loi } \mathcal{E}\text{xp}(1).\end{aligned}$$

Comme $g_k = a_k + e^{-\lambda(t_k - t_{k-1})}(g_{k-1} - a_k)$, on a,

$$\begin{aligned}E_k &= \mathbb{E}[\mathbb{E}(\frac{1}{(1+g_k)^{k+1}}|\mathcal{F}_{k-1})]\\ &= \mathbb{E}[\mathbb{E}(\left(\frac{1}{1+a_k+e^{-\lambda(t_k-t_{k-1})}(g_{k-1}-a_k)}\right)^{k+1}|F_k)]\\ &= \mathbb{E}(\int_0^\infty \left(\frac{1}{1+a_k+(g_{k-1}-a_k)e^{-y/k}}\right)^{k+1} e^{-y}dy), \text{ comme } \lambda k(t_k - t_{k-1}) \text{ est de loi } \mathcal{E}\text{xp}(1).\end{aligned}$$

Remarquons que :

$$\int_0^\infty \left(\frac{1}{1+a+be^{-y/k}})\right)^{k+1} e^{-y}dy = \frac{1}{1+a}(\frac{1}{1+a+b})^k$$

On en déduit la relation de récurrence suivante :

$$E_k = \mathbb{E}(\frac{1}{1+a_k}(\frac{1}{1+g_k})^k) = \frac{1}{1+a_k}E_{k-1}.$$

Il reste à calculer

$$\begin{aligned} E_1 &= \mathbb{E}((\frac{1}{1+g_1})^2) = \mathbb{E}((\frac{1}{1+a_1+e^{-\lambda t_1}(a_0-a_1)})^2) \\ &= \frac{1}{1+a_1}\frac{1}{1+a_0}. \end{aligned}$$

□

1.4 Historique des processus à renforcements linéaires

Depuis l'introduction de la marche renforcée par arête en 1987 par Diaconis [16], beaucoup de travaux sur ce modèle ont été fait. Nous présentons quelques uns de ces travaux, la liste n'est bien sûr pas complète.

Les premiers progrès sont faits par Pemantle sur les arbres réguliers, il a montré une transition de phase en fonction du poids initiaux (a), entre la récurrence positive et la transience.

Theorem 1.4.1 (Pemantle [39]). *Sur un arbre d-régulier ($d \geq 2$), il existe a_c tel que si $a < a_c$, le ERRW avec poids initiaux (a) est récurrent positif et si $a > a_c$, le ERRW est transient.*

Puis Collececchio [14] et Aidekon [1] a donné quelques résultats d'extension, e.g. la loi des grands nombres et le théorème central limite sur des arbres.

L'avantage de travailler sur les arbres est que l'environnement aléatoire se décompose en des urnes de Pólya indépendantes. Si on veux travailler sur les graphes contenant des cycles, par exemple $\mathbb{Z}^d$, l'environnement aléatoire n'est plus indépendant, la question devient plus difficile. Une première approche est due à Merkl et Rolles,

Theorem 1.4.2 (Merkl & Rolles [37]). *Dans un graphe 2-dimensionnel (qui est une version diluée de $\mathbb{Z}^2$), le ERRW est récurrent pour des poids initiaux a suffisamment petits.*

Ce n'est qu'au 2012 que le lien entre le ERRW et le VRJP a été découvert, avant cette date, l'étude du VRJP a été fait de façon séparée. Ces études sont commencés par Davis et Volkov [18] en dimension un, puis sur des arbres [19], ils ont obtenu la récurrence en dimension 1 et la transition de phase sur les arbres :

Theorem 1.4.3 (Davis,Volkov [18, 19]). *(1) Le VRJP sur $\mathbb{Z}$ est récurrent positif pour tous paramètres initiaux constants.*

(2) Le VRJP sur un arbre d-régulier admet une transition de phase en fonction de ses paramètres.

Basdevant et Singh ont généralisé ce résultat sur les arbres de Galton-Watson,

Theorem 1.4.4 (Basdevant & Singh [8]). *Le VRJP sur un arbre de Galton-Watson admet une transition de phase en fonction de ses paramètres et le nombre moyen d'enfants de l'arbre.*

D'ailleurs, Collevecchio a donné une loi des grands nombres et un théorème central limite pour le VRJP sur un arbre d-régulier avec $d \geq 3$.

La conjecture en général est que le ERRW est récurrent dans $\mathbb{Z}^2$ pour tout renforcement, et que dans $\mathbb{Z}^d$ avec $d \geq 3$, il doit y avoir une transition de phase. C'est à dire que, quand les poids a sont suffisamment petits, le ERRW est récurrent, et quand les poids a sont suffisamment grands, le ERRW est transient. Un premier pas important a été fait en 2012 dans cette direction par deux groups de personnes : Sabot, Tarrès et Angel, Crawford, Kozma, par deux approches assez différentes.

Theorem 1.4.5 (Sabot,Tarrès [45], Angel,Crawford,Kozma [4]). *Le ERRW (respectivement le VRJP) dans des graphes de degré borné est récurrent pour des poids (conductance) initiaux suffisamment petits.*

De plus, le résultat de Sabot et Tarrès dévoile aussi le lien entre le ERRW et le VRJP, i.e. le théorème 1.3.8. Une transition de phase a été montré en dimension $d \geq 3$ par Sabot, Tarrès [45] pour le VRJP et par Disertori, Sabot, Tarrès [22] pour le ERRW, en utilisant de façon importante le résultat de délocalisation de Disertori, Spencer, Zirnbauer [23]. Plus précisément, ils utilisent la relation entre la loi du mélange du VRJP et un σ-modèle supersymmetrique. Ce dernier a été introduit et étudié par Disertori, Spencer, Zirnbauer dans [23, 24], où ils ont démontré la localisation à fort renforcement et la delocalization à faible renforcement, qui en fait correspondent respectivement à la récurrence et la transience du VRJP.

Theorem 1.4.6 ([22, 45]). *Sur $\mathbb{Z}^d$ $(d \geq 3)$, le ERRW (respectivement le VRJP) est transient pour des poids initiaux (a) (respectivement des conductances W) constants suffisamment grands.*

D'autre part, Angel,Crawford,Kozma ont montré la transience du ERRW (et du VRJP) pour des faibles renforcements sur des graphes non moyennables [4].

Pour en savoir plus sur les processus à renforcement linéaire, voir le review [38] ; pour les processus avec renforcement en général, voir le review [40].

1.5 Lien avec la diffusion quantique

En 1958, Philip W.Anderson a étudié l'effet des impuretés sur les propriétés de transport électronique dans les cristaux imparfaits. Il a prédit en dimension $d \geq 3$ une transition délocalisée-localisée en fonction de la force du désordre, plus précisément, quand le désordre est suffisamment fort, on doit observer une absence totale de diffusion. Cet effet, dû à des interférences quantiques, est difficile à mettre en évidence expérimentalement, mais aujourd'hui, la localisation forte d'Anderson a été observée dans beaucoup d'expériences, e.g. dans la conductance électronique, dans la transmission des ondes électromagnétiques et des ultrasons.

De façon mathématique, on se place dans l'espace d'Hilbert $\ell^2(\mathbb{Z}^d)$ muni de sa norme ℓ^2 habituelle. Soit Δ l'opérateur Laplacian discrète, i.e. $\Delta : \ell^2(\mathbb{Z})^d \to \ell^2(\mathbb{Z}^d)$ tel que $\forall f \in \ell^2(\mathbb{Z}^d)$

$$(\Delta f)(i) = 2d f(i) - \sum_{j \sim i} f(j).$$

On prend V un champs aléatoire sur $\mathbb{Z}^d$, i.e. $V = (V_i)_{i \in \mathbb{Z}^d}$ est un potentiel aléatoire. Le cas le plus connu est de prendre V_i des variables aléatoires i.i.d. et on note

$$H = \Delta + V$$

où V est considéré comme un opérateur de multiplication. L'opérateur aléatoire H est un opérateur de Schrödinger aléatoire qu'on appelle aussi le modèle d'Anderson.

La question principale concernant la transition localisation-délocalisation qu'on pose sur H est la suivante : Sous quelles propriétés portant sur la loi du potentiel V le spectre de H est purement ponctuel (localisé) ou continu (délocalisé).

Cette question est ouverte depuis 50 ans, en 1991, Zirnbauer [57] a proposé un modèle simplifié qui maintient les caractéristiques du modèle d'Anderson, tel qu'un régime de localisation et de délocalisation peut se démontre en utilisant des techniques de supersymétrie [23, 24].

En 2012, un lien entre ce modèle et le VRJP a été trouvé par Sabot et Tarrès [45]. D'une manière surprenante, ce modèle inventé par une autre communauté est exactement le même que le modèle du VRJP (ainsi ERRW). En particulier, on peut dire que, la formula magique est une conséquence directe de symétrie interne du modèle de Zirnbauer ; qui explique un peu cette formule compliquée et miraculeuse.

Dans le chapitre 5, nous expliquons comment passer du point de vue marche aléatoire à celui du modèle d'Anderson, en particulier, nous construisons un opérateur de Schrödinger aléatoire, et nous montrons que, le comportement de la marche est liée à l'existence de fonction propres généralisées de cet opérateur au bas du spectre.

Il est surprenant que la mesure de mélange du VRJP et le σ-modèle supersymmetrique sont reliés, d'une façon beaucoup plus forte, le fait que la densité (1.4) s'intègre en 1 admet une explication profonde. En fait, en construisant un sigma-modèle supersymétrie, par la Q-symétrie de la construction et le principe de localisation, on 'voit' pourquoi que cette densité est une densité de probabilité. Dans la suite, nous essayons d'expliquer ce point du vue intrigant via un bébé exemple : la surface de la sphère unité est 4π ([53] appendix).

Exemple d'illustration : Q-symétrie et localisation

Soit M la variété $\mathbb{S}^2$, muni de ses coordonnées polaires $0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$, et

$$(x, y, z) = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta).$$

Considérons l'action du groupe de Lie $\mathbb{S}^1$ sur M par rotation $(\phi, \theta) \overset{\varphi_s}{\mapsto} (\phi + s, \theta)$, où $s \in \mathbb{S}^1$, remarquons

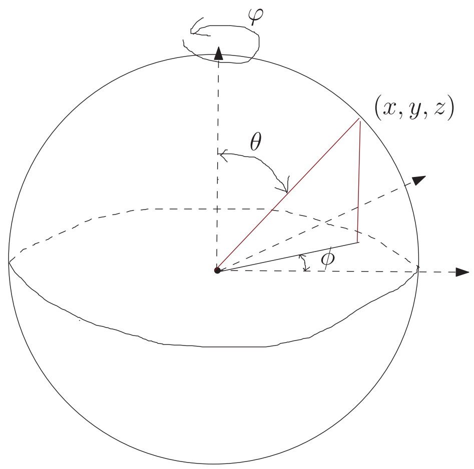

Figure 1.5 : L'action $\varphi : \mathbb{S}^1 \to \mathbb{S}^2$.

que le pôle nord et sud correspondant à $\theta = 0, \pi$ sont les seuls deux points fixes.

Comme le groupe est unidimensionnel, l'action induit un champs de vecteur V dans $\Gamma(TM) = \{V : M \to TM\}$, i.e. $\forall x \in M$,

$$V(x) := \frac{d}{ds}\varphi_s(x)\Big|_{s=0} \in T_x M.$$

La contraction par V est définie comme

$$i_V : \Omega^{k+1}(M) \to \Omega^k(M), \alpha \mapsto i_V(\alpha) = \alpha(V, \cdots)$$

où $\Omega^k(M)$ désigne l'ensemble des k-formes différentielles. On note aussi $\Omega^\star(M) = \bigoplus_{k=0}^n \Omega^k(M)$ (où n est la dimension de variété). On peut ainsi définir l'opérateur différentiel (avec d la différentielle extérieure), pour $\varepsilon > 0$

$$Q := d + \varepsilon i_V : \Omega^\star(M) \to \Omega^\star(M).$$

On a, par la formule de Cartan,

$$Q^2 = \underbrace{d^2}_{=0} + \underbrace{\varepsilon(i_V d + d i_V)}_{L_V \neq 0} + \underbrace{\varepsilon^2 i_V^2}_{=0}$$

où L_V est la dérivée de Lie par rapport à V.

Dans la suite on se restreint à l'ensemble des formes différentielles $\mathbb{S}^1$-invariantes, i.e.

$$\Omega^\star_{\mathbb{S}^1}(M) = \{\alpha \in \Omega^\star(M) | L_V(\alpha) = 0\}$$

car une forme différentielle α est $\mathbb{S}^1$ invariante revient à dire que sa dérivée de Lie est nulle, i.e. $L_V(\alpha) = 0$. On note abusivement $Q = Q_{|\Omega^\star_{\mathbb{S}^1}(M)}$, et on a donc $Q^2 = 0$.

La formule de Duistermaat et Heckman / formule de localisation[2] s'écrit dans ce cadre particulier comme

Proposition 1.5.1. *Soit $\varepsilon > 0$, considérons la forme différentielle $\alpha = \omega - \varepsilon H$ où $\omega = d\phi \wedge dz$ est la métrique habituelle, et $H : M \to \mathbb{R}, (\phi, \theta) \mapsto \cos\theta$ est l'Hamiltonien associé à l'action φ. Alors $\alpha \in Ker(Q)$, et*

$$\int_M e^\alpha = \frac{2\pi}{\varepsilon}(e^\varepsilon - e^{-\varepsilon}).$$

Démonstration. Nous montrons d'abord que $Q(\alpha) = 0$. En fait l'Hamiltonien $H = \cos\theta$ satisfait $i_V\omega - dH = 0$, i.e., V est le champ de vecteur Hamiltonien associé à H, car nous avons

$$i_V(\omega) = i_V(d\phi \wedge dz) = (d\phi \wedge dz)(\partial_\phi, \cdot) = dz = -\sin\theta d\theta = d(\cos\theta) = dH.$$

Donc

$$\begin{aligned} Q(\alpha) &= (d + \varepsilon i_V)(\omega - \varepsilon H) \\ &= \varepsilon(i_V\omega - dH) - \varepsilon^2 \underbrace{i_V H}_{=0} = 0 \end{aligned}$$

Définissons $\eta \in \Omega^1(M)$ par (clairement η est $\mathbb{S}^1$-invariante)

$$\forall u \in T_x M,\ \eta_x(u) =< V(x), u >_g$$

où g est la métrique associée à ω. Montrons que pour tout $t \in \mathbb{R}^+$

$$\int_M e^\alpha = \int_M e^{\alpha - tQ(\eta)}.$$

L'égalité est vraie pour $t = 0$. Par dérivation

$$\begin{aligned} \frac{\partial}{\partial t}\int_M e^{\alpha - tQ(\eta)} &= -\int_M Q(\eta)e^{\alpha - tQ(\eta)} \\ &= -\int_M Q(\eta e^{\alpha - tQ(\eta)}) \\ &= -\underbrace{\int_M d(\eta e^{\alpha - tQ(\eta)})}_{=0 \text{ comme la variété est sans bord}} - \varepsilon \underbrace{\int_M i_V(\eta e^{\alpha - tQ(\eta)})}_{=0 \text{ car } i_V(\cdot) \text{ est d'ordre } <n} = 0. \end{aligned}$$

[2]Au sens où l'intégrale se localise aux points fixes de l'action, i.e. le pôle nord et le pôle sud ici.

où dans la seconde égalité nous avons utilisé $Q(\alpha) = 0,\ Q^2(\eta) = 0$.[3] Ainsi, on a

$$\int_M e^{\alpha} = \int_M e^{\alpha - tQ(\eta)} = \lim_{t\to\infty} \int_M e^{\alpha - td\eta} e^{-\varepsilon t i_V \eta}.$$

Remarquons que

$$(i_V\eta)_x =< V(x), V(x) >_g = |V(x)|_g^2,$$

il vient par définition de g que les seules endroits $|V(x)|_g^2$ s'annulent sont les deux pôles. Si x n'est pas un des deux pôles, on a, soit $B(x,r)$ un voisinage de x sur la sphère, par continuité, pour tout $y \in B(x,r)$, $|V(y)|_g^2$ est borné inférieurement par une constante strictement positive, donc

$$\begin{aligned}\int_{B(x,r)} e^{\alpha} &= \lim_{t\to\infty} \int_{y\in B(x,r)} e^{\alpha} e^{-td\eta} e^{-\varepsilon t|V(y)|_g^2} \\ &= \lim_{t\to\infty} \int_{y\in B(x,r)} e^{-\varepsilon|V(y)|_g^2}(\alpha - td\eta) = 0.\end{aligned}$$

Donc l'intégrale se réduit à intégrer sur les points fixes, si on note $N = (0,0,1),\ S = (0,0,-1)$ les deux pôles, on a,

$$\begin{aligned}\int_{\mathbb{S}^2} e^{\omega - \varepsilon H} &= \left(\int_{B(N,\delta)} e^{\omega-\varepsilon H} + \int_{B(S,\delta)} e^{\omega - \varepsilon H} \right)_{\delta\to 0} \\ &= \left(\int_0^{2\pi} \int_{1-\delta}^{1} e^{-\varepsilon z} dz d\phi + \int_0^{2\pi} \int_{-1}^{-1+\delta} e^{-\varepsilon z} dz d\phi \right)_{\delta\to 0} \\ &= \frac{2\pi}{\varepsilon}(e^{\varepsilon} - e^{-\varepsilon})\end{aligned}$$

Puis en faisant $\varepsilon \to 0$ nous obtenons

$$\int_{\mathbb{S}^2} \omega = \int_{\mathbb{S}^2} e^{\omega} = \lim_{\varepsilon\to 0} \int_{\mathbb{S}^2} e^{\alpha} = 4\pi.$$

□

Un argument similaire en version super mathématique est détaillé dans [23] Appendices, où on considère l'action d'un super groupe de Lie sur certain espace (tel qu'il y ait des fermions), par un argument de localisation de l'intégrale, on est amené à l'égalité suivante : $\int Q_{i_0}^W = 1$. Il n'est pas impossible qu'il y ait d'autres formules qui puissent être trouvées avec cette approche.

1.6 Quelques autres applications de la notion de renforcement

Dans cette section nous donnons quelques applications de la notion de renforcement en général, dans beaucoup de domaines, e.g. la marché publique, le réseau informatique mondial (l'internet), l'étude clinique etc.

[3] en fait Q vérifie la formule de Leibniz en tant qu'un opérateur différentiel

1.6.1 Application 1 : Statistiques bayésienne

Transition de chaîne de Markov

Considérons le problème de statistiques bayésienne suivant : on observe $X_0 = i_0, X_1 = i_1, \dots, X_n = i_n$ un échantillon d'une chaîne de Markov réversible sur un graphe $\mathcal{G}$ fini. Le noyau de transition de cette chaîne de Markov n'est pas connu, pour estimer ce noyau de façon bayésienne, on cherche une famille de distributions sur l'ensemble de noyaux de transition possible sur le graphe $\mathcal{G}$, qui soit stable sous échantillonage. Rappelons que les chaînes réversibles sont exactement des modèles de conductances, donc on cherche une famille de loi $\mu^{(a)}(y_e, e \in E)$ stable par changement de paramètre (a).

Vous devez sûrement remarquer qu'on a un candidat parfait pour cette situation, c'est la formule magique. En [21] Diaconis et Rolles ont montré que l'ensemble des distributions de formule magique est fermé sous l'opération d'échantillonage, et que le posterior après n étapes de la marche, est de paramètre $a_n(e) = a_e + N_n(e)$.

On voit que cette famille de prior généralise celui de Beta, dans le chapitre 5, on introduit une nouvelle famille de distributions, liée à la formule magique, et on espère que cette famille trouve ses applications dans les statistiques bayésiennes.

1.6.2 Application 2 : Part de marché

Limite de la part de marché aléatoire

Supposons que deux produits similaires (l'un n'a pas clairement de l'avantage contre l'autre) se rend dans le marché en même temps, et que les consommateurs choisis l'un des deux avec une préférence proportionelle au nombre de consommateurs qui le possède déjà. C'est un modèle d'urne de Pólya. On sait qu'au final, d'après le Théorème 1.3.1, la part de marché converge vers une variable aléatoire de loi Beta, de plus, si on connaît la part de marché actuelle, le même théorème nous donne la distribution de la part de marché dans le futur. Pour en savoir plus, voir [5].

Monopole aléatoire

Reprenons le modèle précédant, à nouveau on ne suppose pas d'avantage intrinsèque, mais cette fois ci, les clients choisis leur produits à taux proportionelle à une puissance $\alpha > 1$ de la part de marché actuel. Alors, c'est une urne de Pólya généralisée étudiée dans [28], et ça entraîne un marché monopolisé.

1.6.3 Application 3 : Attachement préférentiel

Prenons l'exemple de réseau de citations des articles universitaires, considérons le modèle suivant. On présente les articles déjà publiés comme des points d'un graphe, si un article est cité par une autre, on les relie par une arête. Un nouvelle article sorti va citer exactement m articles déjà publiés, les citations sont choisies de la façon suivante : si on note $d(i)$ le nombre d'articles qui cite l'article i, alors on choisit avec probabilité proportionelle à $d(i)$ l'article i et on cite cet article, puis on le retire dans le graphe et on choisi la citation suivante de la même façon, jusqu'à m articles sont citées.

Cette procédure est appelée l'attachement préférentiel, il peut aussi servir à modéliser le Web, où on simplement remplace les articles par les sites d'internet et les citations par les hyperliens.

Présentons quelques résultats portant sur ce modèle, premièrement, la proportion des sites de degré exactement d est approximativement $\frac{2m^2}{d^3}$, voir [11]. En plus, dans [9], il est démontré que le graphe d'attachement préférentiel converge, au sens de Benjamini-Schramm, vers un graphe appelé 'Pólya point graph'.

1.6.4 Application 4 : Optimisation stochastique

Bandit à deux bras

Pensons au jeu suivant, il y a une machine à deux bras, si on tire le bras gauche, avec probabilité p on gagne une pièce d'un euro ; similairement on gagne un euro avec probabilité q si on tire le bras droit. On ne connaît pas la valeur de p et q, que faut il faire pour maximiser le gain en espérance ?

Une stratégie consiste à faire la chose suivante : prenons une suite $\{\varepsilon_n\}$ convergeant vers 0, à l'instant n, tirer le bras qui a plus de gain moyen empirique avec probabilité $1 - \varepsilon_n$; et tirer l'autre bras avec probabilité ε_n. Dans [25], il est démontré que cette stratégie est asymptotiquement efficace, si on note X_n le gain à l'instant n,

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} X_n = \max\{p, q\}.$$

Étude clinique

Dans une étude clinique on compare deux traitements, l'objectif est de à la fois recueillir de donnés et traiter les patients. Bien évidemment on recueillit le plus de donnés si on prescrivait les deux traitements également souvent, mais dans ce cas-là le traitement moins efficace est prescrit trop souvent.

Une solution utilisant les urnes est la suivante : supposons que ces deux traitements donnent des résultats dichotomies, l'un réussit avec probabilité p et l'autre avec probabilité q, ces probabilités sont inconnues. Imaginons qu'il y a une urne contenant des boules de couleurs rouges et bleues, correspondant aux deux traitements. A chaque instant on tire une boule dans l'urne et prescrit le traitement correspond, si le traitement réussit, on remet α boules de même couleur et $\beta < \alpha$ boules de l'autre couleur ; sinon, on fait l'inverse, i.e. on remet β boules de même couleur et α boules de l'autre couleur.

On peut jouer sur les paramètres α, β afin de trouver une équivalence entre le but de recueillir de donnés et traiter les patients. Par exemple, si on veut minimiser le nombre de prescriptions du traitement inférieur, on peut prendre $\beta = 0$, et on retrouve une urne de Pólya.

1.7 Organisation du reste de la thèse

Le reste du manuscrit est organisé en 5 chapitres. Le chapitre 2 donne une synthèse des résultats obtenus durant la préparation de la thèse, avec quelques idées de preuves. Les chapitres 3 à 6 traitent des questions différentes à propos du VRJP. Plus précisément, le chapitre 3 donne une caractérisation du VRJP en termes de l'échangeabilité partielle et de la dépendance locale de sa probabilité de transition. Le chapitre 4 donne un critère sur la vitesse du VRJP sur les arbres de Galton-Watson. Le chapitre 5 introduit une nouvelle famille exponentielle de mesure généralisant la mesure d'Inverse Gaussian, Nous proposons une représentation du VRJP en termes de cette loi. Le chapitre 6 donne une description de l'environnement aléatoire du VRJP dans un graphe infini, en conséquence, nous montrons un théorème de central limite du VRJP en dimension plus grande que 3 pour des faibles renforcements et la récurrence de la ERRW en dimension deux pour toute valeurs initiales des paramètres.

La dépendance des résultats est montrée dans la figure suivante :

$$\begin{tikzcd}
 & & \text{Le chapitre 3} & \\
\text{Introduction} \arrow[r] & \text{Le chapitre 2} \arrow[ur] \arrow[r] \arrow[dr] & \text{Le chapitre 4} & \\
 & & \text{Le chapitre 5} \arrow[r] & \text{Le chapitre 6}
\end{tikzcd}$$

CHAPTER 2

RESULTS AND IDEAS

In this chapter we present the main results obtained during the preparation of the thesis, some proof ideas are also sketched.

2.1 A characterization of VRJP

In Chapter 3 we give our first result on the VRJP. In the 1920s, Johnson gave a characterization of Pólya urns. Since ERRW is a natural generalization of Pólya urns, in [42], Rolles characterized the ERRW in a similar manner. Our characterization of the VRJP is yet another similar result on the characterization of linearly reinforced processes, where this time we work on continuous time. To give a flavor, before stating the result, we show a similar result which characterizes the Dirichlet's urn, i.e. ERRW on a star; which also is the original idea of Johnson.

Theorem 2.1.1. *Let (X_n) be an exchangeable infinite sequence of random variables valued in $[\![1,\dots,t]\!]$, such that the following conditions hold*

(1) $\forall N \geq 1,\ \forall(\sigma_1,\dots,\sigma_N) \in [\![1,\dots,t]\!]^N,\ \mathbb{P}(X_1 = \sigma_1,\dots,X_N = \sigma_N) > 0.$

(2) $\forall i \in [\![1,\dots,t]\!], \forall N \geq 1$, *there exists functions* $f_i^{(N)} : \mathbb{N} \to [0,1]$ *such that*

$$\mathbb{P}(X_{N+1} = i | X_1,\dots,X_N) = f_i^{(N)}(n_i),$$

where $n_i = \mathrm{card}\{k;\ 1 \leq k \leq N, X_k = i\}$. *Moreover, if* $t = 2$, *we assume that there exists* $a_i^{(N)}, b^{(N)} > 0$ *such that* $f_i^{(N)}(n_i) = a_i^{(N)} + b^{(N)} n_i$.

Then this sequence is an i.i.d. sequence or a t colors Dirichlet's urn, i.e. there exists $k_1,\dots,k_t > 0$, *such that*

$$\mathbb{P}(X_{N+1} = i | X_1,\dots,X_N) = \frac{n_i + k_i}{N + \sum_i k_i}.$$

Proof. Since it is rather elementary, we give the proof of this result in this introduction. Firstly we show that, for any N, if $t \geq 3$, then there exists $a_j^{(N)}, 1 \leq j \leq t, b^{(N)} \in \mathbb{R}$ such that, for any $1 \leq j \leq t$

$$f_j^{(N)}(n) = a_j^{(N)} + b^{(N)} n.$$

If $N \geq 2$, since $t \geq 3$, let $1 \leq j < k < l \leq t$ and

$$\vec{n} = (n_1, \dots, n_j, \dots, n_k, \dots, n_l, \dots, n_t)$$

where $n_1 + \cdots + n_t = N$, by assumption (2) we have

$$\sum_{i=1}^{t} f_i^{(N)}(n_i) = 1. \tag{2.1}$$

Applying (2.1) to

$$\begin{aligned}
\vec{n}(j,k) &= (n_1, \dots, n_j + 1, \dots, n_k - 1, \dots, n_l, \dots, n_t)\\
\vec{n}(l,k) &= (n_1, \dots, n_j, \dots, n_k - 1, \dots, n_l + 1, \dots, n_t)\\
\vec{n}(l,j) &= (n_1, \dots, n_j - 1, \dots, n_k, \dots, n_l + 1, \dots, n_t)
\end{aligned}$$

we deduce that

$$\begin{aligned}
f_j^{(N)}(n_j + 1) - f_j^{(N)}(n_j) &= f_k^{(N)}(n_k) - f_k^{(N)}(n_k - 1)\\
&= f_l^{(N)}(n_l + 1) - f_l^{(N)}(n_l)\\
&= f_j^{(N)}(n_j) - f_j^{(N)}(n_j - 1)
\end{aligned}$$

Therefore, $f_j^{(N)}$ is linear, if we denote $a_j^{(N)} = f_j^{(N)}(0) > 0$ and $b^{(N)} = f_j^{(N)}(1) - f_j^{(N)}(0)$ (this definition does not depend on j), then

$$f_j^{(N)}(n_j) = a_j^{(N)} + b^{(N)} n_j.$$

If $N = 1$, again by (2.1), for any $j \neq k$, $f_j^{(N)}(1) - f_j^{(N)}(0) = f_k^{(N)}(1) - f_k^{(N)}(0)$, the result also holds.

Let us first consider the case that, there exists some N, such that $b^{(N)} = 0$, then, fix some $i \neq j$, by exchangeability (denote $\mathbb{P}(\cdot|(X_1, \dots, X_N) \sim \vec{n}) = \mathbb{P}(\cdot|\vec{n})$)

$$\mathbb{P}(X_{N+1} = i, X_{N+2} = j|\vec{n}) = \mathbb{P}(X_{N+1} = j, X_N = i|\vec{n}). \tag{2.2}$$

That is, $a_i^N(a_j^{(N+1)} + b^{(N+1)} n_j) = a_j^{(N)}(a_i^{(N+1)} + b^{(N+1)} n_i)$. Choose $\vec{n}$ such that $n_i = 0, n_j = N$ and then $n_i = N, n_j = 0$, we deduce that $a_i^{(N)} b^{(N+1)} N = -a_j^{(N)} b^{(N+1)} N$, therefore, $b^{(N+1)} = 0$. Similar argument shows that, if $b^{(N+1)} = 0$, then $b^{(N)} = 0$. As a consequence, $b^{(k)} = 0$ for all k, therefore, (X_n) is an i.i.d. sequence.

Turning to the case $b^{(N)} \neq 0$ for some N, denote $A^{(N)} = \sum_{i=1}^{t} a_i^{(N)}$, we have $A^{(N)} + b^{(N)} N = 1$, denote $K^{(N)} = \frac{A^{(N)}}{b^{(N)}}$ and $k_i^{(N)} = \frac{a_i^{(N)}}{b^{(N)}}$, in particular, we can write

$$f_i^{(N)}(n) = a_i^{(N)} + b^{(N)} n = \frac{n_i + k_i^{(N)}}{N + K^{(N)}}.$$

By (2.2), for any partition $\vec{n}$ of N,

$$k_i^{(N)} n_j + k_j^{(N+1)} n_i + k_i^{(N)} k_j^{(N+1)} = k_i^{(N+1)} n_j + k_j^{(N)} n_i + k_i^{(N+1)} k_j^{(N)} \tag{2.3}$$

Let $n_i = 0, n_j = N$ then let $n_i = N, n_j = 0$. We deduce that $k_i^{(N)} + k_j^{(N)} = k_i^{(N+1)} + k_j^{(N+1)}$. As i, j are arbitrary, hence $K^{(N)} = K^{(N+1)} := K$. If $t > 2$, then we clearly have $k_i^{(N)} = k_i^{(N+1)}$ for any $1 \leq i \leq t$. If $t = 2$, then by taking $n_i = 0, n_j = N$ in (2.3) with $i = 1, j = 2$ we deduce that, $k_1^{(N)}(N + k_2^{(N+1)}) = k_1^{(N+1)}(N + k_2^{(N)})$, therefore, $k_1^{(N)}(N + K^{(N+1)}) = k_1^{(N+1)}(N + K^{(N)})$, that is, $k_1^{(N)} = k_1^{(N+1)}$.

It remains to show that $b^{(N)} > 0$ for all N, since $a_i^{(N)} > 0$, clearly all the $b^{(N)}$ have the same sign, suppose that $b^{(1)} < 0$, then $N + K = \frac{1}{b^{(N)}} < 0$ for all N, which is impossible. □

Rolles' characterization of ERRW

In [42] Theorem 1.1, Rolles proved that if a nearest neighbor random walk is recurrent and partially exchangeable in a reversible sense (c.f. Definition 1.3.1), then it is a mixture of reversible Markov chains.

Rolles' main result in [42] states that, if $G = (V, E)$ is a strongly connected graph and Z_n is a nearest neighbor random walk on G such that the following assumptions are satisfied:

1. Z is partially exchangeable in a reversible sense.
2. For all $v \in V$ and $e \in E$ there exists a function $f_{v,e}$ taking values in $[0, 1]$ such that for all $n \geq 0$

$$\mathbb{P}(Z_{n+1} = v|\mathcal{F}_n) = f_{Z_n,e}(N_{Z_n}(Z_0, \dots, Z_n), \tilde{N}_{Z_n,v}(Z_0, \dots, Z_n)).$$

Then Z is an edge reinforced random walk or a Markov chain under some technical conditions (c.f. [42] for precision).

In Chapter 3, we give a counterpart of the result of Rolles for the VRJP. Let us recall the definition and some features of the VRJP, which will help us to better understand our characterization. Assign positive weights $(W_e)_{e\in E}$ to a finite, connected graph $G = (V, E)$. If $e = \{i, j\}$, then we also write $W_{i,j}$ for W_e. Let Y_t be the VRJP on G, let

$$l_i(t) = \int_0^t \mathbb{1}_{Y_u=i} du,$$

conditionally on the past, Y_t jumps from i to j at rate

$$f_{i,j}(l_j(t)) = W_{i,j}(1 + l_j(t)).$$

Moreover, if $D(s) = \sum_{i\in V}(l_i(s)^2+2l_i(s))$, then $Z_t = Y_{D^{-1}(t)}$ is a mixture of Markov jump processes, c.f.[45] and Chapter 1 Section 1.3.4. As a consequence, Z_t is partially exchangeable, i.e., for each $h > 0$, the law of $\{Z_{nh}, n \geq 1\}$ satisfies the following property: for any two path $\xi = (\xi_0, \dots, \xi_n)$, $\eta = (\eta_0, \dots, \eta_n)$ such that $\xi \sim \eta$, that is, $\xi_0 = \eta_0$ and the transition counts from i to j for any i, j are equal for ξ and η,

$$\mathbb{P}(Z_0 = \xi_0, \dots, Z_{nh} = \xi_n) = \mathbb{P}(Z_0 = \eta_0, \dots, Z_{nh} = \eta_n).$$

An equivalent way to state the partial exchangeability of Z_t consists the following. For any trajectory σ (for convenience, write s_{n+1} for s in the sequel)

$$\sigma := \{Z_{[0,s_1[} = i_0, Z_{[s_1,s_2[} = i_1, \dots, Z_{[s_n,s[} = i_n\}$$

we denote $t_k = s_{k+1}-s_k$. Let $\tilde{Z}_k$ be the discrete time process associated to Z_t and τ_k be the k-th exponential holding time at $\tilde{Z}_k$. Define d_σ as the density of the path σ by the following:

$$\mathbb{P}(\tilde{Z}_0 = i_0, \dots, \tilde{Z}_n = i_n, \tau_0 \in [t_0, t_0 + dt_0], \dots, \tau_{n-1} \in [t_{n-1}, t_{n-1} + dt_{n-1}], \tau_n \geq t_n) \approx d_\sigma \prod_{k=0}^{n} dt_k.$$

If for any σ, d_σ only depends on final local times and transition counts, then the process is partially exchangeable (c.f. Chapter 3, Propositon 3.3.1).

Actually, denoting $S_i(t) = \int_0^t \mathbb{1}_{Z_u=i} du$ the local time of Z, it is computed in [46] that

$$\begin{aligned} d_\sigma =&(\frac{1}{2})^n \prod_{k=1}^{n} W_{i_{k-1},i_k} \prod_{i\in V, i\neq i_n} \frac{1}{\sqrt{1 + S_i(s)}} \\ &\cdot \exp(-\sum_{i\sim j} W_{i,j}(\sqrt{(S_i(s) + 1)(S_j(s) + 1)} - 1)). \end{aligned} \tag{2.4}$$

We observe that this expression only depends on final local times and transition counts, hence we have the partial exchangeability of Z_t. A more detailed study of this expression can be found in Chapter 5. Our characterization theorem is to say that the VRJP is the only partially exchangeable process whose jump rates depend only on the local times of neighboring vertices:

Theorem 2.1.2. *Let X_t be a nearest neighbor jump process on G satisfying the following assumptions:*

1. *For all $i \in V$, there exists C^2 diffeomorphisms h_i such that X is partially exchangeable within the time scale $D(s) = \sum_{i\in V} h_i(l_i(s))$;*
2. *G is strongly connected (i.e. any two adjacent vertices are in a cycle);*
3. *The process, at vertex i at time t, jumps to a neighbor j of i with rate $f_{i,j}(l_j(t))$ for some continuous functions $f_{i,j}$*

Then X is a vertex reinforced jump process within time scale, i.e. there exists another time scale $\tilde{D}$ such that $X_{\tilde{D}^{-1}(t)}$ is a vertex reinforced jump process.

2.2 Speed of VRJP on Galton-Watson trees

Chapter 4 gives another proof of the phase (recurrence/transience) transition of VRJP on a Galton-Watson tree, we also study the speed of VRJP on a Galton-Watson (GW) tree in the transient regime. Recall the result of Sabot and Tarrès, which give the expression of the mixing measure of the VRJP.

Theorem 2.2.1. *On a edge weighed graph $\mathcal{G} = (V, E, W)$, assume V finite. The following measure is a probability distribution on the set $\{(u_i)_{i\in V} \in \mathbb{R}^V,\ u_{i_0} = 0\}$:*

$$\mathcal{Q}_{i_0}^W(du) = \frac{1}{\sqrt{2\pi}^{|V|-1}} \exp\left(-\sum_{i\in V} u_i - \sum_{i\sim j} W_{i,j}(\cosh(u_i - u_j) - 1)\right)\sqrt{D(W,u)}du_{V\setminus\{i_0\}} \tag{2.5}$$

where $du_{V\setminus\{i_0\}} = \prod_{i\in V\setminus\{i_0\}} du_i$ and

$$D(W,u) = \sum_{T\in\mathcal{T}} \prod_{\{i,j\}\in T} W_{i,j} e^{u_i+u_j}$$

The sum is over $\mathcal{T}$, the set of spanning trees of the graph $\mathcal{G}$.

The law of the time changed VRJP (Z_t) starting at i_0 is a mixture of Markov jump processes starting at i_0, with jump rate $\frac{1}{2}W_{i,j}e^{u_j-u_i}$ from i to j, where (u_i) is distributed according to $\mathcal{Q}_{i_0}^W(du)$.

Remark 2.2.1. *As mentioned in Chapter 1, the fact that $\mathcal{Q}_{i_0}^W$ is a probability law was already known since [23].*

If the graph $\mathcal{G}$ is a tree, the probability measure $\mathcal{Q}_{i_0}^W$ takes a much simpler form, it factorized to product of Inverse Gaussian variables. This factorization enable us to reprove the phase transition given by Basdevant and Singh [8].

Consider a supercritical Galton-Watson tree T (without leaf) rooted at ρ with offspring distribution $(q_k, k > 0)$. For some constant $c > 0$, let Y_t be a process starting from ρ with local time $l_i(t)$, and jumps from i to j at rate $c + l_j(t)$ at time t, called the VRJP on T with edge weight $W \equiv 1$ and initial local time c for every vertex, starting a.s. from ρ. Let $D(t) = \sum_{i\in V}(l_i(t)^2 + 2cl_i(t))$, denote VRJP($c$) the time changed process $Z_t = Y_{D^{-1}(t)}$. Moreover, we denote $\overleftarrow{x}$ the parent of vertex x on the tree T, and say that a r.v. A is Inverse Gaussian distributed with parameter $(1, c^2)$ if

$$\mathbb{P}(A \in dx) = \mathbb{1}_{x>0}\frac{c}{2\pi x^3}\exp\{-\frac{c^2(x-1)^2}{2x}\}dx.$$

Theorem 2.2.2. *On a GW tree $T = (V, E)$ rooted at ρ, the time changed VRJP(c) (Z_t) is a mixture of Markov jump processes in i.i.d. random environment $(A_x, x \neq \rho)$, where A_x are i.i.d. Inverse Gaussian random variables with parameter $(1, c^2)$; conditionally on the environment, Z_t jumps at rate*

$$\begin{cases} \frac{1}{2A_x} & \text{from } x \text{ to } \overleftarrow{x} \\ \frac{1}{2}A_x & \text{from } \overleftarrow{x} \text{ to } x. \end{cases}$$

The theorem which we reprove states

Theorem 2.2.3 (Basdevant & Singh). *If $\mu(c) = \mathbb{E}(\sqrt{A})$, then the VRJP$(c)$ on GW tree with offspring mean b is recurrent a.s. if and only if $b\mu(c) \leq 1$.*

Further, we also prove,

Theorem 2.2.4. *Let Z_t be VRJP(c) on a supercritical GW tree such that $b\mu(c) > 1$, we have (denote d the graph distance)*

(1) $v(Z) := \lim_{t\to\infty} \frac{d(\rho, Z_t)}{t}$ exists a.s.

(2) Assume $q_0 = 0$ and $\sum_{k\geq 0} k^2 q_k < \infty$, if $q_1\mathbb{E}(A^{-1/2}) < 1$, then $v(Z) > 0$, if $q_1\mathbb{E}(A^{-1/2}) < 1$, then $v(Z) = 0$.

For the proof, recall that the measure (2.5) factorized into product of independent Inverse Gaussian random variables, we are thus able to consider the VRJP on a tree as a random walk in independent environment.

RWRE on tree are investigated in great details by Hu and Shi, e.g. [30, 29] and Aidekon [1]. In particular, Aidekon have shown a criterion on the positive speed for random walk in site-independent random environment on Galton-Watson trees. Our proof of Theorem 2.2.4 adapts the techniques used in Aidekon's proof, where he firstly seeks for long branches on the GW tree, then compare the random walk to an auxiliary random walk on the half line, with the same type of environment. Thanks to the i.i.d. structure of the environment, he obtains sharp estimates for the one dimensional random walk, which allows him to come back to the tree without losing too much information. This also explains why the criterion depends on q_1, the probability that the GW tree generate one offspring.

However, the random environment of the VRJP is not site-independent, Aidekon's theorem does not apply directly. Since the environment of the VRJP is independent at distance two and has exponential moment, Aidekon's proof ideas still work out with some adaptations.

Figure 2.1 illustrates the traps (long branches) in a GW tree, when the VRJP entered the traps (bold part), it is slowed down. If q_1 is large, there will be many traps in the tree, and the null speed stems from these traps.

2.3 Exponential family related to VRJP

Let us come back to the infinitesimal probability of VRJP, i.e. (2.4), through the window of Theorem 2.2.1. More precisely, the infinitesimal probability of the trajectory

$$\sigma := \{Y_{[0,s_1[} = i_0, Y_{[s_1,s_2[} = i_1, \dots, Y_{[s_n,s[} = i_n\}$$

can be viewed as the annealed version of the quenched infinitesimal probability, that is, the same quantity for the quenched Markov process.

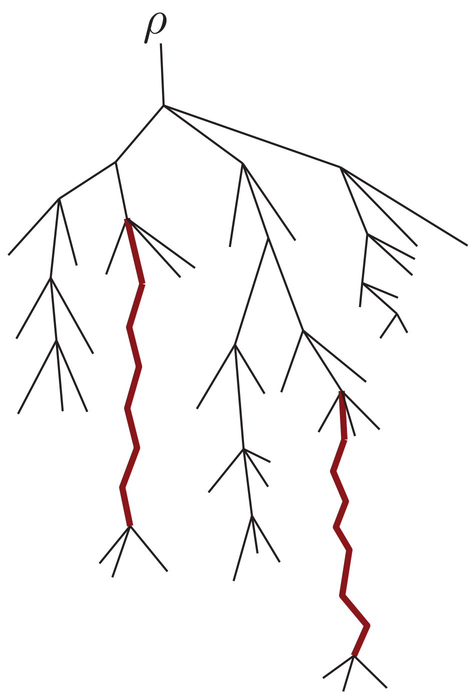

Figure 2.1: Traps on a sampled Galton-Watson tree.

Recall that, when the environment $(u_i)_{i\neq i_0}$ is fixed, the quenched process is Markov with jump rate $\frac{1}{2}W_{i,j}e^{u_j-u_i}$ from i to j. Therefore, the quenched infinitesimal probability for σ (that is, the same quantity as in (2.4) but defined w.r.t. the quenched Markov jump process) is

$$
\begin{aligned}
d_\sigma^{\text{quenched}} &= \frac{1}{2}W_{i_0,i_1}e^{u_{i_1}-u_{i_0}}e^{-\tilde\beta_{i_0}(s_1-s_0)}\cdots\frac{1}{2}W_{i_{n-1},i_n}e^{u_{i_n}-u_{i_{n-1}}}e^{-\tilde\beta_{i_{n-1}}(s_n-s_{n-1})}\cdot e^{-\tilde\beta_{i_n}(s-s_n)}\\
&= (\frac{1}{2})^n\prod_{k=1}^{n}W_{i_{k-1},i_k}e^{u_{i_n}-u_{i_0}}\exp(-\sum_{i\in V}\tilde\beta_i S_i(s))
\end{aligned}
$$

where $\tilde\beta_i$, the rate of the exponential holding time at vertex i is defined by

$$
\tilde\beta_i = \frac{1}{2}\sum_{j\sim i}W_{i,j}e^{u_j-u_i}.
$$

Note that the definition of $\tilde\beta$ only depends on the differences of $u_i, i\in V$, therefore, even if we change to variables $\tilde u_i = u_i - u_{i_0}$ on the domain $\{u_{i_0}=0\}$ of $Q_{i_0}^W$ into $\{\sum_{i\in V}\tilde u_i = 0\}$, $\tilde\beta$ remains the same. The probability $Q_{i_0}^W$ in the new variables $\tilde u$ writes

$$
Q_{i_0}^W(d\tilde u) = \frac{|V|}{\sqrt{2\pi}^{|V|-1}}\exp\left(\tilde u_{i_0} - \sum_{i\sim j}W_{i,j}(\cosh(\tilde u_i-\tilde u_j)-1)\right)\sqrt{D(W,\tilde u)}d\tilde u. \tag{2.6}
$$

Therefore, we see that

$$
\int d_\sigma^{\text{quenched}}(u)dQ_{i_0}^W(du) = d_\sigma
$$

can be rewritten into

$$
\begin{aligned}
&\int \exp(-\sum_{i\in V}\tilde\beta_i S_i(s))dQ_{i_n}^W(d\tilde u)\\
&= \prod_{i\in V, i\neq i_n}\frac{1}{\sqrt{1+S_i(s)}}\cdot\exp(-\sum_{i\sim j}W_{i,j}(\sqrt{(S_i(s)+1)(S_j(s)+1)}-1)).
\end{aligned} \tag{2.7}
$$

If we consider $\tilde{\beta}_i, i \in V$ as random vector distributed with law $\mathcal{Q}^W_{i_n}$, what we have called the infinitesimal probability of a trajectory for the VRJP is in fact the Laplace transform of the random vector $\tilde{\beta}$. Moreover, we observe that, $\tilde{\beta}_i$ are 1-dependent, that is, if $i \nsim j$, then $\tilde{\beta}_i$ and $\tilde{\beta}_j$ are independent. Another important fact is that, we are able to consider $\tilde{\beta}$ as the random environment for the VRJP, since we can compute (u) from $\tilde{\beta}$ by its definition.

We remark that the Laplace transform of $\tilde{\beta}$ depends on the starting point i_0. It is easy to remove this dependence by adding an independent Gamma($\frac{1}{2}$) random variable. Let us define β by

$$\beta = \tilde{\beta} + \mathbb{1}_{i_0}\gamma$$

where γ is Gamma($\frac{1}{2}$) distributed and independent of $\tilde{\beta}$. Clearly the Laplace transform of β has the following form

$$\prod_{i\in V} \frac{1}{\sqrt{1+\lambda_i}} \cdot \exp(-\sum_{i\sim j} W_{i,j}(\sqrt{(\lambda_i+1)(\lambda_j+1)}-1)).$$

The most important result in Chapter 5 is that we explicitly compute the density of β, only using elementary computations. It turns out that, β is an alternative random vector to describe the environment of the VRJP instead of the complicated correlated random vector u. We also discuss some relative consequences in terms of VRJP and ERRW.

Theorem 2.3.1. *Let $\mathcal{G} = (V, E)$ be a (W_e) weighted finite graph as above. The measure below is a probability on $(\mathbb{R}_+)^V$:*

$$\nu^{\mathcal{G},W}(d\beta) := \mathbb{1}_{H_\beta>0} \left(\frac{2}{\pi}\right)^{|V|/2} \exp(-\sum_{i\in V}\beta_i + \sum_{e\in E} W_e)\frac{d\beta_V}{\sqrt{\det H_\beta}} \tag{2.8}$$

with $d\beta_V = \prod_{i\in V} d\beta_i$, and where H_β is the Schrödinger operator on $\mathcal{G}$: $H_\beta = 2\beta - P$ where P is the adjacency matrix of the undirected graph $\mathcal{G}$ with weight (W_e), in other words, H_β is the matrix with coefficients

$$H_\beta(i,j) = \begin{cases} 2\beta_i, & i = j, \\ -W_{i,j}, & i \neq j,\ i\sim j, \\ 0, & \textit{otherwise.} \end{cases}$$

If $(\beta_i, i\in V)$ is $\nu^{\mathcal{G},W}$ distributed, then, the Laplace transform of (β_i) is

$$\mathbb{E}(\exp(-\lambda\cdot\beta)) = \exp\left(-\sum_{i\sim j} W_{i,j}(\sqrt{(\lambda_i+1)(\lambda_j+1)}-1)\right)\prod_{i\in V}\frac{1}{\sqrt{\lambda_i+1}}. \tag{2.9}$$

for all $(\lambda_i) \in \mathbb{R}_+^V$.

Remark 2.3.1. *We see that if we add an independent random gamma variable of parameter $\frac{1}{2}$ to $\tilde{\beta}_{i_n}$,* (2.7) *becomes* (2.9)

It turns out that this random vector (β_i) also gives the random environment of the VRJP on finite graph, moreover, it provides a coupling of the VRJPs starting from different vertices on the same graph $\mathcal{G}$. More precisely, the Green function at i_0, $(G(i_0, i))_{i\in V}$, is the random environment of the VRJP starting from i_0. Moreover, since we have computed the density of β only using elementary computations, this theorem gives a computational proof of $\int \mathcal{Q}^W_{i_0} = 1$, answering a question of Diaconis:

Show that the magic formula is a probability measure by direct computation.

Theorem 2.3.2. *Assume V finite. Let $(\beta_j)_{j\in V}$ be $\nu^{\mathcal{G},W}$ distributed and let $G=(H_\beta)^{-1}$ be the green function of the Schrödinger operator H_β. We denote*

$$e^{u(i,j)}=\frac{G(i,j)}{G(i,i)}. \tag{2.10}$$

For all $i_0\in V$, we have the following properties

(i) the random field $(u(i_0,j))_{j\in V}$ has the distribution $\mathcal{Q}^W_{i_0}$ of Theorem 2.2.1,

(ii) $(u(i_0,j))_{j\in V}$ is $(\beta_j)_{j\in V\setminus\{i_0\}}$-measurable.

(iii) $G(i_0,i_0)$ is equal in law to $\frac{1}{2\gamma}$, where γ is a gamma random variable with parameter $(1/2,1)$,

(iv) $G(i_0,i_0)$ is independent of $(\beta_j)_{j\neq i_0}$, hence independent of the field $(u(i_0,j))_{j\in V}$,

(v) for all $i_0\in V$, $i\in V$

$$\beta_i=\frac{1}{2}\sum_{j\sim i}W_{i,j}e^{u(i_0,j)-u(i_0,i)}+\frac{\mathbb{1}_{i=i_0}}{2G(i_0,i_0)}. \tag{2.11}$$

Remark 2.3.2. *In the functional analysis point of view, on finite graph, the VRJP is the statistical mechanics model associated to the random Schrödinger operator H_β at energy 0.*

More precisely, the adjacency matrix P of $(\mathcal{G},W)$ is the discrete graph Laplacian, one can choose any random potential (typically i.i.d.) to construct a random Schrödinger operator and try to study its spectral property. If we choose the very special measure $\nu^{\mathcal{G},W}(d\beta)$ as our random potential, we have our operator H_β. A common method to study the spectral property of a random Schrödinger operator is to seek statistical mechanics model associated to energy $E\in\mathbb{R}$ (if the operator is not self adjoint, than $E\in\mathbb{C}$), that is, to find probability interpretation of $(H_\beta-E)^{-1}(i_0,\cdot)$.

The VRJP model is the probability model corresponds to the case $E=0$, however for the more interesting cases $E>0$, there is no such corresponding known.

2.4 The random environment of VRJP on infinite graphs

The result of Diaconis, Coppersmith uses in an essential way the fact that the graph is finite (where the random walk in recurrent), a natural question is to ask whether the representation of VRJP as a random walk in random environment still holds on infinite graphs, and try to obtain information of the environment as much as possible. The main theorem of Chapter 6 describes the random environment of VRJP on infinite graphs, we also discuss some consequences of this representation.

Assume from now on that $\mathcal{G}=(V,E,W)$ is infinite, by 1-dependence of β, we can construct a random field β on $\mathcal{G}$ by Kolmogorov's extension theorem.

Proposition 2.4.1. *There exists a family of positive random variables $(\beta_i)_{i\in V}$, such that for any finite subset $U\subset V$, and $(\lambda_i)_{i\in U}\in\mathbb{R}_+^U$*

$$\mathbb{E}\left(e^{-\sum_{i\in U}\lambda_i\beta_i}\right)=e^{-\sum_{i\sim j,\, i,j\in U}W_{i,j}(\sqrt{(1+\lambda_i)(1+\lambda_j)}-1)-\sum_{i\sim j, i\in U, j\notin U}W_{i,j}(\sqrt{1+\lambda_i}-1)}\frac{1}{\prod_{i\in U}\sqrt{1+\lambda_i}}.$$

In particular, $(\beta_i)_{i\in V}$ has the following properties

- *It is 1-dependent : if $U,U'\subset V$ are such that $d_{\mathcal{G}}(U,U')\geq 2$, then $(\beta_i)_{i\in U}$ and $(\beta_j)_{j\in U'}$ are independent.*

- *The marginal β_i is such that $\frac{1}{2\beta_i}$ is an Inverse Gaussian with parameter $(\frac{1}{W_i}, 1)$ where $W_i = \sum_{j\sim i} W_{i,j}$.*

We denote by $\nu^{\mathcal{G},W}(d\beta)$ its distribution.

We can therefore consider the infinite dimensional operator H_β on $\mathcal{G}$, in a similar way as in Theorem 2.3.1. By Remark 2.3.2, to obtain the random environment starting from i_0, $e^{u(i_0,i)} = \frac{G(i_0,i)}{G(i_0,i_0)}$, we again need to compute the inverse G of H_β, but this time it is not even well defined. In fact, some new random variables come into play. Let us first state our main theorem.

Theorem 2.4.1. *Let $\mathcal{G} = (V, E, W)$ be an infinite weighted graph and $(\beta_i, i \in V)$ be the random vector defined in Proposition 2.4.1.*

(1) There exists r.v. $(\hat{G}(i,j), i,j \in V)$ and $(\psi(i), i \in V)$, β-measurable such that

$$\begin{cases} \hat{G} = \lim_{\varepsilon\downarrow 0}(H_\beta + \varepsilon)^{-1} \\ H_\beta \psi = 0 \end{cases}$$

If $\mathcal{G} = \mathbb{Z}^d$ and $W_{i,j} = W$ for all edges $\{i,j\}$, then $(\psi_i)_{i\in V}$ and $(\hat{G}(i,j))_{i,j\in V}$ are stationary and ergodic.

(2) Let γ an independent Gamma variable with parameter $\frac{1}{2}$, define

$$G(i,j) = \hat{G}(i,j) + \frac{1}{2\gamma}\psi(i)\psi(j).$$

Then G is the random environment of the time changed VRJP (Z_t), i.e. if Z_t starts from i_0, then it is a mixture of Markov processes which jump from i to j at rate

$$\frac{1}{2} W_{i,j} \frac{G(i_0, j)}{G(i_0, i)}.$$

Moreover, if the quenched Markov process is transient, then $\psi(i) > 0$ a.s. for all i, otherwise, $\psi(i) = 0$ for all i.

Let us state some consequences of the above representation theorem before we explain the idea of the proof.

Corollary 2.4.1. *Let $(\tilde{Z}_n)_{n\geq 0}$ be the discrete time process associated to VRJP on $\mathbb{Z}^d$, $d \geq 3$, with constant $W_{i,j} = W$. Denote*

$$B_t^{(n)} = \frac{\tilde{Z}_{[nt]}}{\sqrt{n}}.$$

There exists $\lambda_2 > 0$ such that if $W > \lambda_2$, the discrete time VRJP $(\tilde{Z}_n)$ satisfies a functional central limit theorem, i.e. under its law $\mathbb{P}_0^{VRJP}$, $B_t^{(n)}$ converges in law (for the Skorokhod topology) to a d-dimensional Brownian motion B_t with non degenerate isotropic diffusion matrix $\sigma^2 Id$, for some $0 < \sigma^2 < \infty$.

Corollary 2.4.2. *Consider the ERRW $(X_n)_{n\geq 0}$ on $\mathbb{Z}^d$, $d \geq 3$, with constant weights $a_{i,j} = a$. Denote*

$$B_t^{(n)} = \frac{X_{[nt]}}{\sqrt{n}}.$$

There exists $\tilde{\lambda}_2 > 0$ such that if $a > \tilde{\lambda}_2$, the ERRW satisfies a functional central limit theorem, i.e. under its law $\mathbb{P}_0^{ERRW}$, $(B_t^{(n)})$ converges in law (for the Skorokhod topology) to a d-dimensional Brownian motion (B_t) with non degenerate isotropic diffusion matrix $\sigma^2 Id$, for some $0 < \sigma^2 < \infty$.

Corollary 2.4.3. *The ERRW $(X_n)_{n\geq 0}$ on $\mathbb{Z}^2$ with constant weights $a_{i,j} = a$ is a.s. recurrent, i.e.*

$$\mathbb{P}_0^{ERRW}\left(\text{ every vertex is visited infinitely often }\right) = 1.$$

Turning to the strategy of the proof, we construct G as the a.s. limit of some $G^{(n)}$, to be defined later. The wired boundary condition turns out to be handy for this construction.

Definition 2.4.1. *Let $\mathcal{G} = (V, E)$ be a connected graph with finite degree at each site, and V_1 a strict finite subset of V. We define the restriction of $\mathcal{G}$ to V_1 with wired boundary condition as the graph $\mathcal{G}_1 = (\tilde{V}_1 = V_1 \cup \{\delta\}, E_1)$ where δ is an extra point and*

$$E_1 = \{\{i,j\} \in E, \text{ s.t. } i \in V_1, j \in V_1, i \sim j\} \cup \{\{i,\delta\}, i \in V_1 \text{ s.t. } \exists j \notin V_1, i \sim j\}.$$

If $(W_{i,j})_{\{i,j\}\in E}$ is a set of positive conductances, we define $(W_{i,j}^{(1)})_{\{i,j\}\in E_1}$ as the set of restricted conductances by

$$\begin{cases} W_{i,j}^{(1)} = W_{i,j}, & \text{if } i,j \in V_1,\ \{i,j\} \in E_1, \\ W_{i,\delta}^{(1)} = \sum_{j\notin V_1, j\sim i} W_{i,j}, & \text{if } \{i,\delta\} \in E_1, \\ 0, & \text{otherwise.} \end{cases}$$

This restriction corresponds to identify all points in $V \setminus V_1$ to a single point δ and to delete the edges connecting points of $V \setminus V_1$. The new weights are obtained by summing the weights of the edges identified by this procedure.

Now consider (V_n) an increasing sequence of subsets of V which exhausts V, for each n, let $\mathcal{G}_n = (\tilde{V}_n, E_n)$, where $\tilde{V}_n = V_n \cup \{\delta_n\}$ and E_n are defined using the wired boundary conditions. For each $\mathcal{G}_n$, we can associate the so constructed random operator $H_\beta^{(n)}$, denote $G^{(n)}$ the inverse of $H_\beta^{(n)}$.

An important consequence of the wired boundary condition is the following, for any n, since $V_n \subset V_{n+1} \subset V$ are two finite subsets of V, let $(\beta_i^{(n)}, i \in \tilde{V}_n)$, $(\beta_i^{(n+1)}, i \in \tilde{V}_{n+1})$ be the corresponding random vector defined using Theorem 2.3.1, (2.8),

$$\beta_{|V_n}^{(n)} \overset{law}{=} \beta_{|V_n}^{(n+1)}. \tag{2.12}$$

Moreover, we can define a random potential $(\beta_i)_{i\in V}$ and construct a coupling in such a way that

$$\forall n,\ \beta_{|V_n}^{(n)} = \beta_{|V_n},$$

by Theorem 2.3.2 (iii,iv), we are also able to choose γ such that $\forall n,\ G^{(n)}(\delta_n, \delta_n) = \frac{1}{2\gamma}$ where γ is a Gamma r.v. with parameter $\frac{1}{2}$, independent of β.

In terms of density, (2.12) writes

$$\int f^{(n+1)}(\beta) \prod_{i\in\tilde{V}_{n+1}\setminus V_n} d\beta_i = \int f^{(n)}(\beta) d\beta_{\delta_n} \tag{2.13}$$

where $f^{(n+1)}$ and $f^{(n)}$ are the density function defined with (2.8). The most important observation in Chapter 6 is that, differentiating[1] w.r.t. $W_{\delta,j}$ in (2.13) yields that,

Proposition 2.4.2. *With the coupling constructed as above, for any $j \in V$,*

$$\psi^{(n)}(j) := \frac{G^{(n)}(\delta_n, j)}{G^{(n)}(\delta_n, \delta_n)}$$

is a positive $\sigma\{\beta_i, i \in V_n\}$ martingale, in particular it converges a.s.

[1] Actually, there are several technicalities to this issue, but we omit them in this introduction.

Remark 2.4.1. *Note that* $\psi^{(n)}(j) = e^{u^{(n)}(\delta_n,j)}$ *where* $e^{u^{(n)}(\delta_n,j)}$ *is defined in* (2.10) *for* $H_\beta^{(n)}$ *on the graph* $\mathcal{G}_n$*, the mixing field of the VRJP starting from the point* δ_n*.*

Our limiting procedure follows from the following key observation:

$$G^{(n)}(i,j) = \hat{G}^{(n)}(i,j) + G^{(n)}(\delta_n,\delta_n)\psi^{(n)}(i)\psi^{(n)}(j) \tag{2.14}$$

where

$$\hat{G}^{(n)} = (H_\beta^{(n)})^{-1}_{|V_n}.$$

Remark 2.4.2. *The decomposition* (2.14) *is actually the so called random walk expansion at* δ_n *of* $H_\beta^{(n)}$*, which is a consequence of resolvant identity. That is, if we look at the* (i,j) *coordinate of the following equality*

$$\frac{1}{H-T} = \frac{1}{H} + \frac{1}{H}T\frac{1}{H} + \frac{1}{H}T\frac{1}{H-T}T\frac{1}{H}$$

where $H := (H_\beta^{(n)})_{|V_n}$ *is the operator restricted to* V_n *and* T *is the boundary effect of* $H_\beta^{(n)}$*, i.e.* $H_\beta^{(n)} = H - T$*, then we obtained* (2.14).

Therefore, we can pass to limit in (2.14), since we have $\hat{G}^{(n)}(i,j)$ is increasing, $G^{(n)}(\delta_n,\delta_n)$ is constant, and $\psi^{(n)}(i)$, $\psi^{(n)}(j)$ are both martingales. Our main theorem then follows.

In terms of the infinite dimensional random Schrödinger operator H_β, we actually give an explicit construction of a generalized eigenfunction at energy 0, that is, the bottom of the spectrum. This generalized eigenfunction is the limit of the finite dimensional ground state at δ, that is, the ground state of $H_\beta^{(n)}$ at δ_n converges for every vertex j. This feature is possibly not true in the case of i.i.d. potential, in some sense it explains why it is suggested that the $H^{(2|2)}$-model (i.e. the toy model invented in order to study Anderson localizantion, which corresponds to the VRJP, see [23]) is easier to analyze than the original Anderson localization model.

CHAPTER 3

A CHARACTERIZATION OF VRJP

(based on the paper 'How vertex reinforced jump process arises naturally' published in AIHP) [55]

Abstract

We prove that the only nearest neighbor jump process with local dependence on the occupation times satisfying the partially exchangeable property is the vertex reinforced jump process, under some technical conditions (Theorem 3.2.2). This result gives a counterpart to the characterization of edge reinforced random walk given by Rolles [42].

Keywords: Partial exchangeability, Vertex reinforced jump processes.

3.1 Introduction

One of the most remarkable results in probabilistic symmetries is the de Finetti's theorem [], which states that the law of any exchangeable sequence valued in a finite state space is in fact a mixture of i.i.d. sequences. This theorem has a geometrical interpretation via Choquet's theorem. More precisely, the subspace of exchangeable probabilities forms a convex, and those probabilities given by i.i.d. sequences are exactly the extreme points of the convex [3].

In the 1920s, W.E. Johnson [54] conjectured that, under some technical conditions, if a process X_n is exchangeable and $\mathbb{P}(X_{n+1} = i | X_0, \ldots, X_n)$ depends only on the number of times i occurs and the total steps n, then X_n is nothing but the famous Polya urn: drawing balls uniformly from an urn and put back one additional ball with same color as the drawn one. This is a process with linear reinforcement. In term of random walk, the natural counterpart of Polya urn is the edge reinforced random walk (ERRW). Diaconis conjectured that this process have the same characterization as Polya urn. In [42] S.W.W.Rolles have shown that both conjectures are true under technical conditions.

The vertex reinforced jump process (VRJP) is a linearly reinforced process in continuous time. In a recent paper, Sabot and Tarres [45] have shown that ERRW is a mixture of VRJP, which indicates that the VRJP are building blocks of ERRW, thus should share a similar characterization. This paper gives this characterization (Theorem 3.2.2), as a counterpart of Rolles' result; namely, the only continuous time process which is partially exchangeable and the transition probability depends only on neighbor local times is VRJP, under technical conditions.

Let us first recall the definition of ERRW, let $G = (V, E)$ be a locally finite undirected graph without direct loops (edges with one endpoint). Let Z_n denote the location of the random process at time n. Let

$a_e > 0$, $e \in E$. For $n \in \mathbb{N}$, define $w_n(e)$, the weight of edge e at time n, by

$$w_0(e) = a_e \text{ for all } e \in E\ ,$$
$$w_{n+1}(e) = \begin{cases} w_n(e) + 1 & \text{for } e = \{Z_n, Z_{n+1}\} \in E, \\ w_n(e) & \text{for } e \in E \setminus \{\{Z_n, Z_{n+1}\}\}. \end{cases}$$

Let $\mathbb{P}_{v_0}^{(a)}$ denote the probability of the ERRW on G starting at v_0 with initial weights $a = (a_e)_{e \in E}$. Then $\mathbb{P}_{v_0}^{(a)}$ is defined by

$$Z_0 = v_0,\ \mathbb{P}_{v_0}^{(a)} - a.s.,$$
$$\mathbb{P}_{v_0}^{(a)}(Z_{n+1} = v | Z_0, \dots, Z_n) = \begin{cases} \frac{w_n(\{Z_n, v\})}{\sum_{e, Z_n \in e} w_n(e)} & \text{if } \{Z_n, v\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

Now let us introduce some definitions before stating Rolles' result. Again $G = (V, E)$ is a locally finite undirected graph without direct loops, with its vertex set V and edge set E. Denote $i \sim j$ if $\{i, j\} \in E$. Following Rolles, we call $(Z_n)_{n \geq 0}$ a nearest neighbor random walk on G, if it is a discrete time random process (not necessarily Markov) such that successive positions are neighbors.

An admissible path of the random walk is a sequence of vertices of G, denoted $\pi = (v_0, v_1, \dots, v_n)$ such that consecutive vertices are neighbors. The number of visits to vertex i of path π is denoted

$$N_i(\pi) := \#\{k : \ v_k = i,\ k = 0, \dots, n\};$$

Similarly, the number of transition counts in the path π of an oriented edge $e = (i, j)$ is denoted

$$N_e(\pi) = N_{i,j}(\pi) := \#\{k : \ v_k = i, v_{k+1} = j,\ k = 0, \dots, n-1\}.$$

Two paths ξ, η are said to be equivalent and denoted $\xi \sim \eta$, if ξ and η start at the same state and the transition counts from i to j of any pair (i, j) are equal for ξ and η, i.e. $N_{i,j}(\xi) = N_{i,j}(\eta)$ for all (i, j).

Remark 3.1.1. *Two equivalent paths necessarily end at the same vertex.*

Definition 3.1.1. *A nearest neighbor random walk is partially exchangeable if any two equivalent paths have the same probability.*

Theorem 3.1.1 (Diaconis & Freedman [20]). *Let Z_n be a recurrent random walk (i.e. with probability one it returns to Z_0 infinitely often), then Z is a mixture of Markov chains if and only if it is partially exchangeable. Moreover, the mixing measure is uniquely determined.*

As it turns out that edge reinforced random walk is a mixture of reversible Markov chains, Rolles introduced the following more restrictive notion of partial exchangeability: for $\pi = (v_0, \dots, v_n)$ and $e = (i, j)$ let

$$\tilde{N}_e(\pi) := \#\{k : \ v_k = i, v_{k+1} = j \text{ or } v_k = j, v_{k+1} = i,\ k = 0, \dots, n-1\}.$$

Definition 3.1.2. *A nearest neighbor random walk is partially exchangeable in a reversible sense if it satisfies the following: for any two paths ξ, η, if $\tilde{N}_e(\xi) = \tilde{N}_e(\eta)$ for all $e \in E$, then ξ and η have the same probability.*

In [42] Theorem 1.1, Rolles proved that if a nearest neighbor random walk is recurrent and partially exchangeable in a reversible sense, then it is a mixture of reversible Markov chain.

Rolles' main result in [42] states that, if $G = (V, E)$ is a strongly connected graph and Z_n is a nearest neighbor random walk on G such that the following assumptions are satisfied:

1. Z is partially exchangeable in a reversible sense (Definition 3.1.2).

2. For all $v \in V$ and $e \in E$ there exists a function $f_{v,e}$ taking values in $[0, 1]$ such that for all $n \geq 0$

$$\mathbb{P}(Z_{n+1} = v|\mathcal{F}_n) = f_{Z_n,e}(N_{Z_n}(Z_0, \dots, Z_n), \tilde{N}_{Z_n,v}(Z_0, \dots, Z_n)).$$

Then Z is an edge reinforced random walk or a Markov chain under some technical conditions (c.f. [42] for precision).

Next we define the vertex reinforced jump process X_t. Assign positive weights $(W_e)_{e\in E}$ to the edges, the process X_t starts at time 0 at some vertex i_0, if X is at vertex $i \in V$ at time t, then, conditioned on the past, the process jumps to a neighbor j of i with rate $W_{i,j}(1 + l_j(t))$, where for $e = \{i, j\}$, $W_{i,j} = W_e$ and $l_j(t)$ is the local time of vertex j at time t:

$$l_j(t) := \int_0^t \mathbb{1}_{X_s=j} ds.$$

Theorem 3.1.2 (Sabot & Tarres[45]). *The ERRW Z_n with weights (a_e) is equal in law to the discrete time process associated with a VRJP X_t in random independent weights $W_e \sim$* Gamma$(a_e, 1)$

And finally, the VRJP X_t turns out to be partially exchangeable within a time scale (c.f. next section for the definition of partial exchangeability in continuous times). Let

$$D(s) = \sum_{i\in V}(l_i(s)^2 + 2l_i(s)),$$

then the process $Y_t = X_{D^{-1}(t)}$ is a mixture of Markov processes with an explicit mixing measure, in addition, the mixing measure turns out to be related to a σ-model introduced by Zirnbauer, c.f. [45] Theorem 2.

In this paper we give a counterpart of Rolles' result for VRJP, namely we characterize exchangeable jump processes with local rate functions.

3.2 Definitions and results

Definition 3.2.1. *We call $(X_t)_{t\geq 0}$ a nearest neighbor jump process on G, if it is a random process which is right continuous without explosion, and each jump is from some vertex i to one of its neighbors j (i.e. $i \sim j$).*

Definition 3.2.2. *A nearest neighbor jump process X_t is a mixture of Markov jump processes if there exists a probability measure μ on Markov jump processes such that $\mathcal{L}(X_t) = \int \mathcal{L}(Y_t)\mu(dY)$, where $\mathcal{L}$ denotes the law of respective processes. If for μ a.s. the Markov processes are reversible, then the process X_t is a mixture of reversible Markov processes.*

Freedman introduced the notion of partial exchangeability in continuous time in [27].

Definition 3.2.3 (Freedman). *A continuous process X_t is partially exchangeable if for each $h > 0$, the law of $\{X_{nh}; n = 1, 2, \cdots\}$ satisfies the following property: for any two paths $\xi = (\xi_0, \dots, \xi_l)$, $\eta = (\eta_0, \dots, \eta_l)$ such that $\xi \sim \eta$,*

$$\mathbb{P}(X_0 = \xi_0, \dots, X_{lh} = \xi_l) = \mathbb{P}(X_0 = \eta_0, \dots, X_{lh} = \eta_l).$$

We recall the de Finetti's theorem in continuous time showed by Freedman [27].

Theorem 3.2.1. *Let X_t be a continuous time process starting at $i_0 \in G$, X_t is mixture of Markov jump processes if*

1. *X_t has no fixed points of discontinuity, more precisely, for every t, if $t_n \to t$, then $\mathbb{P}(X_{t_n} \to X_t) = 1$;*

2. *X_t is recurrent;*

3. *X_t is partially exchangeable.*

Our main theorem is:

Theorem 3.2.2. *Let X_t be a nearest neighbor jump process on G satisfying the following assumptions:*

1. *For all $i \in V$, there exists C^2 diffeomorphisms h_i such that X is partially exchangeable within the time scale $D(s) = \sum_{i \in V} h_i(l_i(s))$;*

2. *G is strongly connected (i.e. any two adjacent vertices are in a cycle);*

3. *The process, at vertex i at time t, jumps to a neighbor j of i with rate $f_{i,j}(l_j(t))$ for some continuous functions $f_{i,j}$*

Then X is a vertex reinforced jump process within time scale, i.e. there exists another time scale $\tilde{D}$ such that $X_{\tilde{D}^{-1}(t)}$ is a vertex reinforced jump process.

Remark 3.2.1. *In fact, the hypothesis of Theorem 3.2.2 implies that the functions $f_{i,j}(x)$ are necessarily of the form $W_{i,j}x + \varphi_j$.*

Remark 3.2.2. *Note that we do not a priori require $f_{i,j} = f_{j,i}$, i.e. there is no assumption of reversibility for X_t; however the VRJP is a mixture of reversible Markov jump processes within time change.*

Remark 3.2.3. *Concerning the third assumption, we cannot prove the result with rate $f_{i,j}(l_i, l_j)$, but the case where $f_{i,j}(l_i, l_j) = f_i(l_i) f_j(l_j)$ can be treated. In fact, by applying a time change, the process with rate function of the form $f_i(l_i) f_j(l_j)$ can be reduced to our theorem.*

In section 3, we introduce an equivalent notion of partial exchangeability and, as an example, we give a different proof of partial exchangeability of VRJP within a time scale. Section 4 contains the proof of Theorem 3.2.2.

3.3 The two notions of partial exchangeability

3.3.1 Partial exchangeability, infinitesimal point of view

Consider a nearest neighbor jump process on G satisfying the third assumption of Theorem 3.2.2. As we have assumed regularity on the trajectory of the process (c.f. Definition 3.2.1), to describe the law of our process, it is enough to describe the probability of the following events:

$$\sigma = \{X_{[0,t_1[} = i_0, X_{[t_1,t_2[} = i_1, X_{[t_2,t_3[} = i_2, \dots, X_{[t_{n-1},t_n[} = i_{n-1}, X_{[t_n,t]} = i_n\},$$

which will be denoted

$$\sigma : i_0 \xrightarrow{t_1} i_1 \xrightarrow{t_2 - t_1} i_2 \dots i_{n-1} \xrightarrow{t_n - t_{n-1}} i_n \xrightarrow{t - t_n}$$

in the sequel and we call such an event *a trajectory*.

It turns out that when the jump rate is a continuous function of local times, the law of our process can be characterized by some function, which will be called *density* in the sequel. In fact, for the study of certain history depending random processes, we have the following lemma:

Lemma 3.3.1. *If (X_t) is a jump process with jump rate depending only on local times and the current position of the random walker, i.e. there exists functions $f_{i,j}(l)$ such that conditioned on the past, X_t jumps from i to j at rate $f_{i,j}(l(t))$, and, moreover, $f_{i,j}(l(t))$ does not depend on the variable $l_i(t)$. Then there exists functions d_σ, such that for all bounded measurable functions Φ defined on the trajectories,*

$$\mathbb{E}(\Phi(X_u,\ u \le t)) = \sum_{n\ge 1} \sum_{i_0,\ldots,i_n} \int d_\sigma \Phi(\sigma) dt + d_{i_0 \xrightarrow{t}} \Phi(i_0 \xrightarrow{t})$$

where $d_\sigma = \exp(-\int_0^t \sum_{j\sim X_s} f_{X_s,j}(l(s))ds) \prod_{k=1}^n f_{i_{k-1},i_k}(l(t_k))$ *and* $d_{i_0 \xrightarrow{t}} = \mathbb{P}(X_s = i_0, 0 \le s \le t)$.

Remark 3.3.1. *We believe that Lemma 3.3.1 still hold when $f_{i,j}(l)$ depends on $l_i(t)$. In fact, if we can find a time changed process such that its jump rates do not depend on $l_i(t)$, it is immediate by re-applying the inverse time change that Lemma 3.3.1 holds in the general cases.*

Proof. As $f_{i,j}(l(t))$ does not depend on $l_i(t)$, the holding time of X_t at i is exponentially distributed with rate

$$\sum_{j\sim i} f_{i,j}(l(t))$$

and the probability of jumping from i to j is

$$p(i,j) := \frac{f_{i,j}(l(t))}{\sum_{k\sim i} f_{i,k}(l(t))}.$$

Moreover, the process up to time t is characterized by the events

$$i_0 \xrightarrow{s_1} i_1 \xrightarrow{s_2} \cdots \xrightarrow{s_n} i_n \xrightarrow{s_{n+1}},\ s_1, \ldots, s_{n+1} > 0, \sum_{i=1}^{n+1} s_i \le t.$$

For $1 \le k \le n+1$, denote $t_k = s_1 + \cdots + s_k$,

$$\begin{aligned}
&\mathbb{P}(X_t \text{ follows the trajectory } i_0 \xrightarrow{s_1} i_1 \xrightarrow{s_2} \cdots \xrightarrow{s_n} i_n \xrightarrow{s_{n+1}},\ s_k > 0,\ \sum s_k \le t) \\
&= \int_{t_n \le t} \prod_{k=1}^n \left(p(i_{k-1}, i_k) \exp(-\sum_{j\sim i_{k-1}} f_{i_{k-1},j}(l(t_{k-1}))s_k) \cdot \sum_{j\sim i_{k-1}} f_{i_{k-1},j}(l(t_{k-1})) \right) \mathbb{P}(s_{n+1} > t - t_n) ds \\
&= \int_{t_1<t_2<\cdots<t_n<t} \exp(-\int_0^t \sum_{j\sim X_s} f_{X_s,j}(l(s))ds) \prod_{k=1}^n f_{i_{k-1},i_k}(l(t_{k-1}))dt,
\end{aligned}$$

with $ds = ds_1 \cdots ds_n$, $dt = dt_1 \cdots dt_n$. Now the lemma follows by distinguishing different trajectories. □

Definition 3.3.1. *We say that X_t admits a density if the assumptions in Lemma 3.3.1 are satisfied, and we denote its density as d_σ.*

Let us now give another definition of partial exchangeability for continuous time processes in terms of density. Define two trajectories σ and τ to be equivalent and denoted $\sigma \sim \tau$, if their discrete chain strings are equivalent and the local times are equal at each vertex. Formally,

Definition 3.3.2. *Let*

$$\sigma = i_0 \xrightarrow{t_1} i_1 \xrightarrow{t_2-t_1} i_2 \cdots i_{n-1} \xrightarrow{t_n-t_{n-1}} i_n \xrightarrow{t-t_n},$$

$$\tau = j_0 \xrightarrow{s_1} j_1 \xrightarrow{s_2-s_1} j_2 \cdots j_{n-1} \xrightarrow{s_n-s_{n-1}} j_n \xrightarrow{t-s_n} .$$

Then σ and τ are equivalent if and only if

$$\begin{cases} \forall i \in V,\ l_i^\sigma(t) = l_i^\tau(t) \\ \forall i,j\ N_{i,j}(\sigma) = N_{i,j}(\tau). \end{cases}$$

where $N_{i,j}(\sigma)$ denotes the number of jumps from i to j in σ, i.e. $N_{i,j}(\sigma) = N_{i,j}((i_0,\dots,i_n))$, and $l_i^\sigma(t) = \int_0^t \mathbb{1}_{\sigma_s=i} ds$ denotes the local time.

Definition 3.3.3. *A continuous time nearest neighbor jump process is said to be partially exchangeable in density if the densities are equal for any two equivalent trajectories. Or equivalently, the density depends only on final local times and the transition counts.*

3.3.2 Equivalence of the two notions

It turns out that in the case of nearest neighbor jump process with continuous jump rate functions, the notion of partial exchangeability in Definition 3.2.3 and in Definition 3.3.3 are equivalent.

Proposition 3.3.1. *If a continuous time nearest neighbor jump process is partially exchangeable in the sense of Definition 3.3.3, then it is partially exchangeable in the sense of Definition 3.2.3.*

Proof. Suppose that the process X_t is partially exchangeable in density, let $h > 0$, consider the event $I = \{X_0 = i_0, X_h = i_1, \dots, X_{nh} = i_n\}$, let $(j_0 = i_0, j_1, \dots, j_n)$ be an equivalent string of $(i_0,\dots,i_n)$, and $J = \{X_0 = j_0, X_h = j_1, \dots, X_{nh} = j_n\}$.

We construct a bijection T which maps trajectories of I to those of J. As $(i_0,\dots,i_n)$, $(j_0,\dots,j_n)$ are equivalent, for any pair of neighbors (i,j), there are exactly a same number of transition counts from i to j. Let us define T to be the transformation which is a permutation of the time segmentations $[lh,(l+1)h)$ of size h; which, for any k, moves the kth transition $i \xrightarrow{k\text{th}} j$ of I to the kth transition $i \xrightarrow{k\text{th}} j$ of J, and leaving the last time segmentation $[nh,\infty)$ invariant. Figure 3.1 illustrates an example of such application.

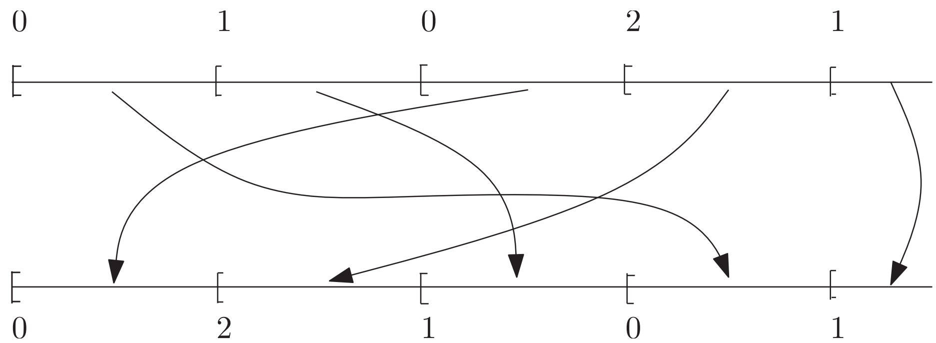

Figure 3.1: The transformation T for $I = \{X_0 = 0, X_h = 1, X_{2h} = 0, X_{3h} = 2, X_{4h} = 1\}$ and $J = \{X_0 = 0, X_h = 2, X_{2h} = 1, X_{3h} = 0, X_{4h} = 1\}$.

Let

$$\sigma = k_0 \xrightarrow{s_1} k_1 \xrightarrow{s_2} k_2 \cdots k_{N-1} \xrightarrow{s_N} k_N \xrightarrow{s_{N+1}}$$

be one trajectory of the event I, we check that

$$T(\sigma) = k'_0 \xrightarrow{s'_1} k'_1 \xrightarrow{s'_2} k'_2 \cdots k'_{N-1} \xrightarrow{s'_N} k'_N \xrightarrow{s'_{N+1}}$$

is a trajectory of the event J, and that T is one-one and on-to (c.f. Figure 3.2). If we fix the total number of jumps N and the discrete trajectory $(k_0, k_1, \ldots, k_N)$, then T can be though as a substitution of integration. Thus

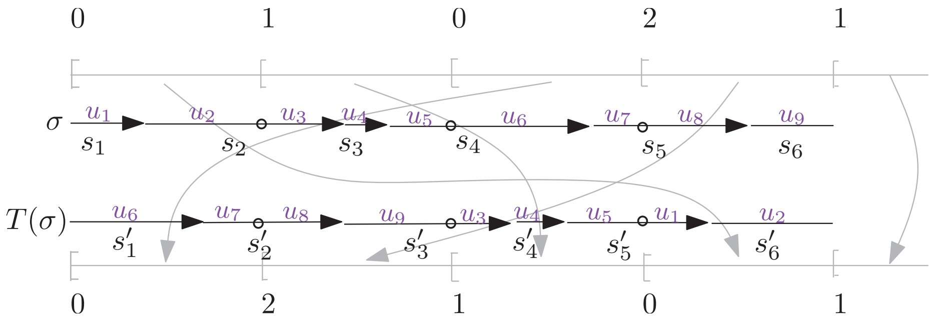

Figure 3.2: An example of σ and $T(\sigma)$.

$$\begin{aligned}\mathbb{P}(I) &= \sum_N \sum_{k_0,k_1,\ldots,k_N} \int \mathbb{1}_{s_1,\ldots,s_{N+1} \in I(N,k_0,\ldots,k_N)} d_\sigma ds_1 \cdots ds_{N+1} \\ &= \sum_N \sum_{k'_0,k'_1,\ldots,k'_N} \int \mathbb{1}_{s'_1,\ldots,s'_{N+1} \in I'(N,k'_0,\ldots,k'_N)} d_{T(\sigma)} ds'_1 \cdots ds'_{N+1} = \mathbb{P}(J),\end{aligned}$$

where $I(N, k_0, \ldots, k_N)$ is the subset of $\mathbb{R}^{N+1}$ defined as the set of $(s_1, \ldots, s_{N+1})$ such that the event $k_0 \xrightarrow{s_1} k_1 \xrightarrow{s_2} \cdots k_N \xrightarrow{s_{N+1}}$ is in I; and $I'(N, k'_0, \ldots, k'_N)$ is its image by applying T; see Figure 2 for a concrete example. As T preserves local times and the numbers of transition counts, these two integrals are whence equal. □

Proposition 3.3.2. *If a jump process is partially exchangeable in the sense of Definition 3.2.3, and its jump rate is a continuous function of local times, then it is also partially exchangeable in the sense of Definition 3.3.3.*

Proof. Let X_t denote such a process, for $h > 0$, consider the σ-algebra $\mathcal{F}_h = \sigma(X_{nh}, n \geq 0)$, let

$$\mathcal{F}_0 = \sigma(\cup_{h>0} \mathcal{F}_h)$$

and

$$\mathcal{F} = \sigma(X_t, t \geq 0).$$

As in [27], we only consider h running through the binary rationals. Note that $\mathcal{F}_0 = \mathcal{F}$ thanks to the right continuity of the trajectories.

Let $\sigma = i_0 \xrightarrow{t_1} i_1 \xrightarrow{t_2 - t_1} i_2 \cdots i_n \xrightarrow{t - t_n}$ be a trajectory with n jumps (say $n \geq 1$ to avoid triviality). Let $\{X^{(h)} \sim \sigma/h\}$ denotes the event

$$\{X_0 = \sigma_0,\ X_h = \sigma_h, \ldots, X_{Nh} = \sigma_{Nh}, \quad \text{with } N = \lfloor t/h \rfloor\}.$$

It turns out that

$$d_\sigma = \lim_{h \to 0} \mathbb{P}(X^{(h)} \sim \sigma/h) h^{-n}.$$

In fact, let $\Psi = \mathbb{1}_{X^{(h)}\sim\sigma/h}$, by definition of d_σ,

$$\mathbb{E}(\Psi(X_u,\ u \leq t)) = \mathbb{P}(X^{(h)} \sim \sigma/h) = \sum_{k\geq 1}\sum_{i_1,\ldots,i_k} \int d_\tau \Psi(\tau) dt_1 \cdots dt_k \tag{3.1}$$

where

$$\tau = i_0 \xrightarrow{t_1} i_1 \xrightarrow{t_2-t_1} i_2 \cdots i_{k-1} \xrightarrow{t_k-t_{k-1}} i_k \xrightarrow{t-t_k} .$$

When h is small enough, the sum in (3.1) must be over $k \geq n$, and we have

$$\mathbb{P}(X^{(h)} \sim \sigma/h) = \mathbb{P}_1 + \mathbb{P}_2.$$

where for some $p_k, k = 1, \ldots, n$ depending on h

$$\begin{aligned}
\mathbb{P}_1 &= \mathbb{P}((X_u)_{0\leq u\leq t} \text{ makes } n \text{ jumps at times } s_1, \ldots, s_n \\
&\qquad \text{with } s_k \in (p_k h, (p_k+1)h] \text{ and the trajectory is } i_0, \ldots, i_n) \\
\mathbb{P}_2 &= \mathbb{P}((X_u)_{0\leq u\leq t} \text{ makes more than } n+1 \text{ jumps and } X^{(h)} \sim \sigma/h)
\end{aligned}$$

Note that the jump rates are bounded from both below and above, and any holding time in the event of $\mathbb{P}_2$ must be in an interval of length lesser than $2h$, whence the probability of making $n+l$ ($l \geq 1$) jumps following the trajectory σ/h is smaller than the probability of $n+l$ independent exponential variables (of parameter C) each smaller than $2h$, where C is an upper bound of the jump rates. Whence

$$\mathbb{P}_2 \leq \sum_{l\geq 1} (\mathbb{P}(\text{cst} \leq \mathcal{E}xp(C) < \text{cst} + 2h))^{n+l} \leq \sum_{l\geq 1} (\mathbb{P}(\mathcal{E}xp(C) < 2h))^{n+l} = O(h^{n+1}).$$

Thus $\mathbb{P}_2$ can be dropped when taking the limit. In addition,

$$\mathbb{P}_1 = \int_{p_n h}^{(p_n+1)h} \cdots \int_{p_1 h}^{(p_1+1)h} d_\sigma\, dt_1 \cdots dt_n,$$

note that here d_σ depends only on $t_1, \ldots, t_n$ and it is an absolutely integrable function, by Lebesgue differentiation theorem (Theorem 1.6.19 [49]) $\lim_{h\to 0} \mathbb{P}_1/h^n = d_\sigma$. Now let $\sigma \sim \tau$, when h is sufficiently small,

$$\begin{array}{ccc}
d_\sigma & \longleftarrow & \mathbb{P}(X_{ih} \sim \sigma/h)h^{-n} \\
 & & \| \\
d_\tau & \longleftarrow & \mathbb{P}(X_{ih} \sim \tau/h)h^{-n}
\end{array}$$

proceeding as in the diagram shows that $d_\sigma = d_\tau$. □

3.3.3 Example: VRJP is partially exchangeable within a time change

Recall that $Y_s = X_{D^{-1}(s)}$ with $D(s) = \sum_{i\in V}(l_i(s)^2 + 2l_i(s))$, It turns out that we can write down the density of the trajectory σ of the (time changed) VRJP process Y (For convenience, write s_{n+1} for s in the sequel). The density of

$$\sigma := i_0 \xrightarrow{s_1} i_1 \xrightarrow{s_2-s_1} i_2 \cdots i_{n-1} \xrightarrow{s_n-s_{n-1}} i_n \xrightarrow{s-s_n}$$

is (c.f. [46]), denoting $S_i(t) = \int_0^t \mathbb{1}_{Y_u=i} du$ the local time of Y,

$$
\begin{aligned}
d_\sigma =& (\frac{1}{2})^n \prod_{k=1}^n W_{i_{k-1},i_k} \prod_{i\in V, i\neq i_n} \frac{1}{\sqrt{1+S_i(s)}} \\
&\cdot \exp(-\sum_{i\sim j} W_{i,j}(\sqrt{(S_i(s)+1)(S_j(s)+1)}-1)),
\end{aligned} \tag{3.2}
$$

which clearly depends only on final local times and transition counts, thus by Proposition 3.3.1, Y is partially exchangeable. On finite graph it is rather easy to prove that the VRJP is recurrent (for example, using a representation of VRJP by time changed Poisson point process as in [45], and then use an argument as in [17] or [48]). Therefore, Y is a mixture of Markov jump processes.

For convenient, we include a proof of this in the sequel (after the proof of Proposition 3.4.1), since the mechanisms of the proof enlightens the proof of the main theorem.

3.4 Proof of the characterization theorem

3.4.1 Computation of densities

Let X be a nearest neighbor jump process on G satisfying the assumptions of Theorem 3.2.2, in particular, recall the time scale

$$
D(s) = \sum_{i\in V} h_i(l_i(s)). \tag{3.3}
$$

Let $l_i(t)$ be the local time of the process X at vertex i at time t. Let us denote the process after time change to be

$$
Y_t = X_{D^{-1}(t)}, \tag{3.4}
$$

let

$$
S_i(s) = \int_0^s \mathbb{1}_{Y_u=i} du \tag{3.5}
$$

denote the local time of Y. Consider the trajectory

$$
\sigma : i_0 \xrightarrow{t_1} i_1 \xrightarrow{t_2-t_1} i_2 \cdots i_{n-1} \xrightarrow{t_n-t_{n-1}} i_n \xrightarrow{t-t_n} \tag{3.6}
$$

where $0 < t_1 < \cdots < t_n < t$, after applying the time change, the corresponding trajectory for Y is

$$
\sigma_Y : i_0 \xrightarrow{s_1} i_1 \xrightarrow{s_2-s_1} i_2 \cdots i_{n-1} \xrightarrow{s_n-s_{n-1}} i_n \xrightarrow{s-s_n}
$$

where $s_k = D(t_k)$.

Proposition 3.4.1. *With the same settings as in equations* (3.3) (3.4) (3.5) (3.6), *the density of the trajectory* σ_Y *for* Y *is*

$$
d_\sigma^Y = \exp\left(-\int_0^s \sum_{j\sim Y_v} \frac{f_{Y_v,j}(h_j^{-1}(S_j(v)))}{h'_{Y_v}(h_{Y_v}^{-1}(S_{Y_v}(v)))} dv\right) \prod_{k=1}^n \frac{f_{i_{k-1},i_k}(h_{i_k}^{-1}(S_{i_k}(s_{k-1})))}{h'_{i_{k-1}}(h_{i_{k-1}}^{-1}(S_{i_{k-1}}(s_k)))}.
$$

Proof. Applying Lemma 3.3.1 to the process X,

$$
d_\sigma = \exp\left(-\int_0^t \sum_{j\sim X_u} f_{X_u,j}(l_j(u)) du\right) \prod_{k=1}^n f_{i_{k-1},i_k}(l_{i_k}(t_{k-1})).
$$

Recall that in (3.3) we assumed that $h_i : \mathbb{R}^+ \to \mathbb{R}^+$ are diffeomorphisms satisfying $h_i(0) = 0$.

Next we compute the density for the same trajectory σ but for the process $Y_s = X_{D^{-1}(s)}$, as we have $S_i(D(s)) = h_i(l_i(s))$, derivation leads to

$$S_i(D(s))' = D'(s)\mathbb{1}_{Y_{D(s)}=i} = h_i'(l_i(s))\mathbb{1}_{X_s=i}.$$

Hence

$$(D^{-1}(t))' = \frac{1}{D'(D^{-1}(t))} = \frac{1}{h'_{Y_{t-}} \circ h^{-1}_{Y_{t-}}(S_{Y_{t-}}(t))},$$

$$l_{i_k}(t_{k-1}) = h^{-1}_{i_k}(S_{i_k}(D(t_{k-1}))) = h^{-1}_{Y_{s_k}}(S_{Y_{s_k}}(s_{k-1})).$$

Substituting $s = D(t)$, we have

$$d^Y_\sigma = \exp\left(-\int_0^s \sum_{j\sim Y_v} \frac{f_{Y_v,j}(h_j^{-1}(S_j(v)))}{h'_{Y_v}(h^{-1}_{Y_v}(S_{Y_v}(v)))} dv\right) \prod_{k=1}^n \frac{f_{i_{k-1},i_k}(h^{-1}_{i_k}(S_{i_k}(s_{k-1})))}{h'_{i_{k-1}}(h^{-1}_{i_{k-1}}(S_{i_{k-1}}(s_k)))}.$$

□

Back to the partial exchangeability of VRJP

Proof of (3.2). Apply the previous proposition to VRJP, where $f_{i,j}(l_j) = W_{i,j}(1 + l_j)$ and $h_i(l_i) = l_i^2 + 2l_i$. The density d^Y_σ is

$$\frac{1}{2^n} \exp\left(-\int_0^s \sum_{j\sim Y_u} \frac{W_{Y_u,j}\sqrt{S_j(u)+1}}{2\sqrt{S_{Y_u}(u)+1}} du\right) \prod_{k=1}^n \left(W_{i_{k-1},i_k} \frac{\sqrt{S_{i_k}(s_{k-1})+1}}{\sqrt{S_{i_{k-1}}(s_k)+1}}\right).$$

As our trajectory is left continuous without explosion, starting at i_0, if we calculate the product through the trajectory, by telescopic simplification, it results that the product reduces to

$$\prod_{i\in V\ i\neq i_n} \frac{1}{\sqrt{S_i(s)+1}} \prod_{k=1}^n W_{i_{k-1},i_k}.$$

To compute the integral inside the exponential, it is enough to note that, in the expression:

$$\sum_{i\sim j} W_{i,j}(\sqrt{(S_i(s)+1)(S_j(s)+1)} - 1),$$

the local times $S_i(s), i \in V$ of the process Y only vary (linearly) with s when the process is at i, i.e., when $Y_t = i$. Therefore, the derivative of the above expression with respect to s equals to

$$\sum_{j\sim Y_s} \frac{W_{Y_s,j}\sqrt{S_j(s)+1}}{2\sqrt{S_{Y_s}(s)+1}}$$

which is what we integrate inside the exponential.

Whence (3.2) is proved, and expression (3.2) depends only on final local times and transition counts, the result hence follows. □

3.4.2 Determination of time change h

In the sequel we work with the time changed process Y, to simplify notations, we will write d_σ for d_σ^Y when it does not lead to any confusion. By Proposition 3.4.1, the density of certain trajectory contains an exponential term and a product term, let us denote

$$d_\sigma = \exp(-\int \sigma) \cdot \prod \sigma,$$

with

$$\begin{cases} \int \sigma = \int_0^s \sum_{j\sim Y_v} \frac{f_{Y_v,j}(h_j^{-1}(S_j(v)))}{h'_{Y_v}(h_{Y_v}^{-1}(S_{Y_v}(v)))} dv \\ \prod \sigma = \prod_{k=1}^n \frac{f_{i_{k-1},i_k}(h_{i_k}^{-1}(S_{Y_{s_k}}(s_{k-1})))}{h'_{i_{k-1}}(h_{i_{k-1}}^{-1}(S_{Y_{s_{k-1}}}(s_k)))} \end{cases}$$

where the exponential term stems from those exponential waiting times, and the product term corresponds to the probability of the discrete chain.

The heuristics of the proof in this subsection is the following: as we assumed partial exchangeability, if we consider two equivalent trajectories, then their densities share the same expression, by comparing them we can hence deduce certain equalities involving $f_{i,j}$ and h_i etc. It turns out that these equalities determine h_is then $f_{i,j}$s.

The following fact is simple but important, suppose that at time s, the random walker arrives at i_0, each vertex i has accumulated local time $l_i := S_i(s)$; then it jumps to i_1 after an amount of time t, by Proposition 3.4.1, the density has acquired a multiplicative factor

$$\exp\left(-\int_s^{s+t} \sum_{j\sim i_0} \frac{f_{i_0,j}\circ h_j^{-1}(l_j)}{h'_{i_0}\circ h_{i_0}^{-1}(l_{i_0}+v)} dv\right) \cdot \frac{f_{i_0,i_1}\circ h_{i_1}^{-1}(l_{i_1})}{h'_{i_0}\circ h_{i_0}^{-1}(l_{i_0}+t)}. \tag{3.7}$$

This fact is in constant use in the sequel, when we explicit the density of certain trajectory.

Lemma 3.4.1. *Let $\sigma = i_0 \xrightarrow{s_1} i_1 \xrightarrow{s_2-s_1} i_2 \cdots i_{n-1} \xrightarrow{s_n-s_{n-1}} i_n \xrightarrow{s-s_n}$ be a trajectory, then $\int \sigma = \int \tilde{\sigma} + \int \hat{\sigma}$ where*

$$\int \tilde{\sigma} = \int_0^s \sum_{j\in\sigma, j\sim Y_v} \frac{f_{Y_v,j}(h_j^{-1}(S_j(v)))}{h'_{Y_v}(h_{Y_v}^{-1}(S_{Y_v}(v)))} dv, \quad \int \hat{\sigma} = \int_0^s \sum_{j\notin\sigma, j\sim Y_v} \frac{f_{Y_v,j}(h_j^{-1}(S_j(v)))}{h'_{Y_v}(h_{Y_v}^{-1}(S_{Y_v}(v)))} dv$$

and if τ is such that $\tau \sim \sigma$, then $\int \hat{\sigma} = \int \hat{\tau}$.

Proof. Note that for $j \notin \sigma$, $S_j(u) = 0$ for all $u \le s$. Let $\hat{H}_i$ be the primitive of $\frac{1}{h'_i \circ h_i^{-1}}$ such that $\hat{H}_i(0) = 0$,

$$\begin{aligned} \int \hat{\sigma} &= \sum_{j\notin\sigma} \int_0^s \mathbb{1}_{Y_v\sim j} \frac{f_{Y_v,j}(0)}{h'_{Y_v}(h_{Y_v}^{-1}(S_{Y_v}(v)))} dv \\ &= \sum_{j\notin\sigma, i\in\sigma, j\sim i} f_{i,j}(0) \int_0^s \frac{\mathbb{1}_{Y_v=i}}{h'_i(h_i^{-1}(S_i(v)))} dv \\ &= \sum_{j\notin\sigma, i\in\sigma, j\sim i} f_{i,j}(0) \hat{H}_i(S_i(s)) \end{aligned}$$

which depends only on final local times, thus if $\tau \sim \sigma$, then $\int \hat{\tau} = \int \hat{\sigma}$. □

In the sequel cst denotes some constant, which can vary from line to line.

Lemma 3.4.2. *If the process X admits such a time change D which makes it partially exchangeable in density, then for any $i \sim j$, there exists some constants $\lambda_{i,j}$ such that*

$$f_{i,j}(x) = \lambda_{i,j} h_j'(x),\ \forall x \geq 0. \tag{3.8}$$

Proof. Let $\epsilon > 0$, consider the following two trajectories for the process Y:

$$\sigma = i \xrightarrow{\epsilon} j \xrightarrow{\epsilon} i \xrightarrow{t} j \xrightarrow{s} i \xrightarrow{\cdot}$$

$$\tau = i \xrightarrow{t} j \xrightarrow{s} i \xrightarrow{\epsilon} j \xrightarrow{\epsilon} i \xrightarrow{\cdot}$$

Note that σ and τ have the same transition counts and the final local times on vertex i, j are respectively equal. Thus the densities of these trajectories are a.s. equal by partial exchangeability. By Lemma 3.4.1,

$$d_\sigma = \prod \sigma \cdot \exp(\int \tilde{\sigma} + \int \hat{\sigma}),$$

where

$$\begin{cases} \prod \sigma = \frac{f_{i,j} \circ h_j^{-1}(0)}{h_i' \circ h_i^{-1}(\epsilon)} \cdot \frac{f_{j,i} \circ h_i^{-1}(\epsilon)}{h_j' \circ h_j^{-1}(\epsilon)} \cdot \frac{f_{i,j} \circ h_j^{-1}(\epsilon)}{h_i' \circ h_i^{-1}(\epsilon+t)} \cdot \frac{f_{j,i} \circ h_i^{-1}(\epsilon+t)}{h_j' \circ h_j^{-1}(\epsilon+s)} \\ \int \tilde{\sigma} = \int_0^\epsilon \frac{f_{i,j} \circ h_j^{-1}(0)}{h_i' \circ h_i^{-1}(v)} dv + \int_0^\epsilon \frac{f_{j,i} \circ h_i^{-1}(\epsilon)}{h_j' \circ h_j^{-1}(v)} dv + \int_0^t \frac{f_{i,j} \circ h_j^{-1}(\epsilon)}{h_i' \circ h_i^{-1}(\epsilon+v)} dv + \int_0^s \frac{f_{j,i} \circ h_i^{-1}(\epsilon+t)}{h_j' \circ h_j^{-1}(\epsilon+v)} dv. \end{cases}$$

$$d_\tau = \prod \tau \cdot \exp(\int \tilde{\tau} + \int \hat{\tau}),$$

where

$$\begin{cases} \prod \tau = \frac{f_{i,j} \circ h_j^{-1}(0)}{h_i' \circ h_i^{-1}(t)} \cdot \frac{f_{j,i} \circ h_i^{-1}(t)}{h_j' \circ h_j^{-1}(s)} \cdot \frac{f_{i,j} \circ h_j^{-1}(s)}{h_i' \circ h_i^{-1}(t+\epsilon)} \cdot \frac{f_{j,i} \circ h_i^{-1}(\epsilon+t)}{h_j' \circ h_j^{-1}(\epsilon+s)} \\ \int \tilde{\tau} = \int_0^t \frac{f_{i,j} \circ h_j^{-1}(0)}{h_i' \circ h_i^{-1}(v)} dv + \int_0^s \frac{f_{j,i} \circ h_i^{-1}(t)}{h_j' \circ h_j^{-1}(v)} dv + \int_0^\epsilon \frac{f_{i,j} \circ h_j^{-1}(s)}{h_i' \circ h_i^{-1}(t+v)} dv + \int_0^\epsilon \frac{f_{j,i} \circ h_i^{-1}(\epsilon+t)}{h_j' \circ h_j^{-1}(s+v)} dv; \end{cases}$$

We do not explicit $\int \hat{\sigma}$ and $\int \hat{\tau}$ as they cancel when we compare these expressions (c.f. Lemma 3.4.1).

Letting $\epsilon \to 0$ yields that $\exp(\int \tilde{\sigma}) = \exp(\int \tilde{\tau})$; therefore $\prod \sigma = \prod \tau$, i.e.

$$\forall s, t,\ \frac{f_{i,j} \circ h_j^{-1}(s)}{h_j' \circ h_j^{-1}(s)} \cdot \frac{f_{j,i} \circ h_i^{-1}(t)}{h_i' \circ h_i^{-1}(t)} = cst.$$

Now fix t, let s vary, whence

$$\forall s,\ f_{i,j} \circ h_j^{-1}(s) = cst \cdot h_j' \circ h_j^{-1}(s),$$

and let $\lambda_{i,j}$ denotes this constant, as h_j^{-1} is a diffeomorphism, its range is $\mathbb{R}^+$, which allows us to conclude. □

The next lemma states in some sense that the exponential part and the product part appearing in the density of a trajectory can be treated separately.

Lemma 3.4.3. *Let σ, τ be two trajectories, and denote*

$$d_\sigma = \exp(\int \sigma) \cdot \prod \sigma, \quad d_\tau = \exp(\int \tau) \cdot \prod \tau,$$

if $\sigma \sim \tau$, then $\prod \sigma = \prod \tau$.

Proof. We have $S_{Y_{s_k}}(s_k) = S_{Y_{s_k}}(s_{k-1})$, thus Lemma 3.4.2 yields that $f_{i_{k-1},i_k} \circ h_{i_k}^{-1}(S_{Y_{s_k}}(s_{k-1})) = \lambda_{i_{k-1},i_k} h'_{i_k} \circ h_{i_k}^{-1}(S_{Y_{s_k}}(s_k))$. Whence the product part is

$$\prod \sigma = \prod_{k=1}^{n} \frac{f_{i_{k-1},i_k}(h_{i_k}^{-1}(S_{Y_{s_k}}(s_{k-1})))}{h'_{i_{k-1}}(h_{i_{k-1}}^{-1}(S_{Y_{s_{k-1}}}(s_k)))} = \prod_{k=1}^{n} \lambda_{i_{k-1},i_k} \frac{\prod_{i\neq i_0} h'_i \circ h_i^{-1}(0)}{\prod_{i\neq i_n} h'_i \circ h_i^{-1}(S_i(s))},$$

and the last term depends only on the transition counts and final local times. □

Lemma 3.4.4. *Let $H_i = h'_i \circ h_i^{-1}$, then for some constant A_i (recall that h_i is assumed C^2 diffeomorphism),*

$$(H_i^2)' = A_i \text{ and if } i \sim j, \text{ then } \lambda_{i,j} A_j = \lambda_{j,i} A_i.$$

Remark 3.4.1. *The latest equality tells that the process is reversible. However, we did not assume the reversibility of the process, but vertex reinforced jump processes are reversible (as a mixture of reversible Markov jump process), so are the edge reinforced random walks. In contrast, directed edge reinforced random walks are mixtures of non reversible Markov chains, with independent Dirichlet environments. We can hence expect that the reversibility is a consequence of a non oriented linear reinforcement (where linearity leads to partial exchangeability).*

Proof. Recall that we have assumed that the graph is strongly connected, i.e. if i, j are two adjacent vertices, there exists a shortest cycle $i_1 \sim i_2 \sim i_3 \cdots \sim i_n \sim i_1$ with $i_1 = i, i_n = j$ and the i_ks are distinct and $n \geq 2$.

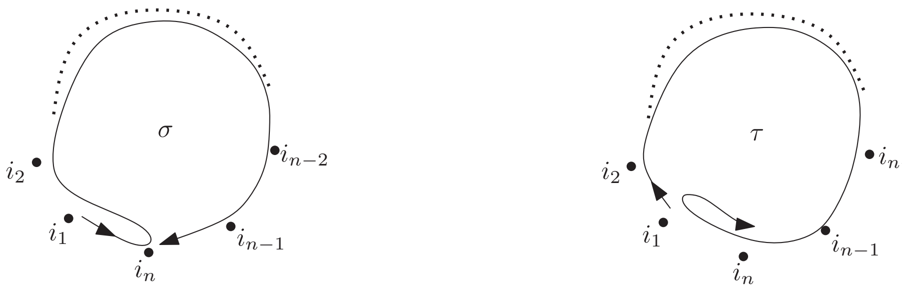

Figure 3.3: the trajectories σ and τ in Lemma 3.4.4.

Let $(i_1 = i, i_2, i_3, \ldots, i_n = j)$ be a cycle as described, consider the trajectories (c.f. Figure 3.3)

$$\sigma = i_1 \xrightarrow{r_1} i_n \xrightarrow{r_2} i_1 \xrightarrow{s_1} i_2 \xrightarrow{s_2} i_3 \cdots i_{n-2} \xrightarrow{s_{n-2}} i_{n-1} \xrightarrow{s_{n-1}} i_n$$

$$\tau = i_1 \xrightarrow{r_1} i_2 \xrightarrow{s_2} i_3 \cdots i_{n-2} \xrightarrow{s_{n-2}} i_{n-1} \xrightarrow{s_{n-1}} i_n \xrightarrow{r_2} i_1 \xrightarrow{s_1} i_n.$$

As $\sigma \sim \tau$, by Lemma 3.4.3 and Lemma 3.4.1, $\int \tilde{\sigma} = \int \tilde{\tau}$. Also let

$$\sigma' = i_1 \xrightarrow{r_1} i_n \xrightarrow{r_2} i_1 \xrightarrow{s_1} i_2 \xrightarrow{s_2} i_1$$

$$\tau' = i_1 \xrightarrow{r_1} i_2 \xrightarrow{s_2} i_1 \xrightarrow{s_1} i_n \xrightarrow{r_2} i_1,$$

thus $\int \tilde{\sigma}' = \int \tilde{\tau}'$. We are going to compute explicitly $\int \tilde{\sigma}$, $\int \tilde{\tau}$ etc, using (3.7), let $s = r_1+r_2+s_1+\cdots+s_{n-1}$

and recall that $\hat{H}_i$ is the primitive of $\dfrac{1}{h_i'\circ h_i^{-1}}$ such that $\hat{H}_i(0)=0$.

$$\begin{aligned}
\int \tilde{\sigma} &= \sum_{(i,j)\in\sigma^2, i\sim j} \lambda_{i,j} \int_0^s \mathbb{1}_{Y_v=i} \frac{h_j'\circ h_j^{-1}(S_j(v))}{h_i'\circ h_i^{-1}(S_i(v))} dv \\
&= \lambda_{i_1,i_2} H_{i_2}(0)\hat{H}_{i_1}(r_1+s_1) + \lambda_{i_2,i_1} H_{i_1}(r_1+s_1)\hat{H}_{i_2}(s_2) \\
&+ \lambda_{i_1,i_n}\left(H_{i_n}(0)\hat{H}_{i_1}(r_1) + H_{i_n}(r_2)(\hat{H}_{i_1}(r_1+s_1) - \hat{H}_{i_1}(r_1))\right) \\
&+ \lambda_{i_n,i_1} H_{i_1}(r_1)\hat{H}_{i_n}(r_2) + \lambda_{i_n,i_{n-1}} H_{i_{n-1}}(0)\hat{H}_{i_n}(r_2) \\
&+ \lambda_{i_{n-1},i_n} H_{i_n}(r_2)\hat{H}_{i_{n-1}}(s_{n-1}) + \Delta
\end{aligned}$$

where Δ is defined as follows: let $Q_k := H_{i_k}(0)\hat{H}_{i_{k-1}}(s_{i_{k-1}})$ and $Q_k' := H_{i_k}(s_k)\hat{H}_{i_{k+1}}(s_{i_{k+1}})$,

$$\Delta = \sum_{k=3}^{n-1} \lambda_{i_{k-1},i_k} Q_k + \lambda_{i_k,i_{k-1}} Q_{k-1}'.$$

For $\tilde{\tau}$ we have:

$$\begin{aligned}
\int \tilde{\tau} &= \sum_{(i,j)\in\tau^2, i\sim j} \lambda_{i,j} \int_0^s \mathbb{1}_{Y_v=i} \frac{h_j'\circ h_j^{-1}(S_j(v))}{h_i'\circ h_i^{-1}(S_i(v))} dv \\
&= \lambda_{i_1,i_2} H_{i_2}(0)\hat{H}_{i_1}(r_1) + H_{i_2}(s_2)(\hat{H}_{i_1}(r_1+s_1) - \hat{H}_{i_1}(r_1)) \\
&+ \lambda_{i_2,i_1} H_{i_1}(r_1)\hat{H}_{i_2}(s_2) \\
&+ \lambda_{i_1,i_n}\left(H_{i_n}(0)\hat{H}_{i_1}(r_1) + H_{i_n}(r_2)(\hat{H}_{i_1}(r_1+s_1) - \hat{H}_{i_1}(r_1))\right) \\
&+ \lambda_{i_n,i_1} H_{i_1}(r_1)\hat{H}_{i_n}(r_2) + \lambda_{i_n,i_{n-1}} H_{i_{n-1}}(s_{n-1})\hat{H}_{i_n}(r_2) \\
&+ \lambda_{i_{n-1},i_n} H_{i_n}(0)\hat{H}_{i_{n-1}}(s_{n-1}) + \Delta
\end{aligned}$$

with the same Δ. Also

$$\begin{aligned}
\int \tilde{\sigma}' &= \lambda_{i_1,i_2}\left(H_{i_2}(0)\hat{H}_{i_1}(r_1) + H_{i_2}(0)(\hat{H}_{i_1}(r_1+s_1) - \hat{H}_{i_1}(r_1))\right) \\
&+ \lambda_{i_2,i_1} H_{i_1}(r_1+s_1)\hat{H}_{i_2}(s_2) \\
&+ \lambda_{i_1,i_n}\left(H_{i_n}(0)\hat{H}_{i_1}(r_1) + H_{i_n}(r_2)(\hat{H}_{i_1}(r_1+s_1) - \hat{H}_{i_1}(r_1))\right) \\
&+ \lambda_{i_n,i_1} H_{i_1}(r_1)\hat{H}_{i_n}(r_2)
\end{aligned}$$

$$\begin{aligned}
\int \tilde{\tau}' &= \lambda_{i_1,i_2}\left(H_{i_2}(0)\hat{H}_{i_1}(r_1) + H_{i_2}(s_2)(\hat{H}_{i_1}(r_1+s_1) - \hat{H}_{i_1}(r_1))\right) \\
&+ \lambda_{i_2,i_1} H_{i_1}(r_1)\hat{H}_{i_2}(s_2) \\
&+ \lambda_{i_1,i_n}\left(H_{i_n}(0)\hat{H}_{i_1}(r_1) + H_{i_n}(0)(\hat{H}_{i_1}(r_1+s_1) - \hat{H}_{i_1}(r_1))\right) \\
&+ \lambda_{i_n,i_1} H_{i_1}(r_1+s_1)\hat{H}_{i_n}(r_2).
\end{aligned}$$

Recall that $\int \tilde{\sigma} - \int \tilde{\sigma}' = \int \tilde{\tau} - \int \tilde{\tau}'$, which leads to

$$\begin{aligned}
&\lambda_{i_n,i_{n-1}} H_{i_{n-1}}(0)\hat{H}_{i_n}(r_2) + \lambda_{i_{n-1},i_n} H_{i_n}(r_2)\hat{H}_{i_{n-1}}(s_{n-1}) \\
&= \lambda_{i_1,i_n}(H_{i_n}(r_2) - H_{i_n}(0))(\hat{H}_{i_1}(r_1+s_1) - \hat{H}_{i_1}(r_1)) \\
&+ \lambda_{i_n,i_1}(H_{i_1}(r_1) - H_{i_1}(r_1+s_1))\hat{H}_{i_n}(r_2) \\
&+ \lambda_{i_n,i_{n-1}} H_{i_{n-1}}(s_{n-1})\hat{H}_{i_n}(r_2) + \lambda_{i_{n-1},i_n} H_{i_n}(0)\hat{H}_{i_{n-1}}(s_{n-1})
\end{aligned}$$

letting $s_{n-1} \to 0$ leads to

$$\lambda_{i_1,i_n}(H_{i_n}(r_2) - H_{i_n}(0))(\hat{H}_{i_1}(r_1 + s_1) - \hat{H}_{i_1}(r_1)) =$$
$$\lambda_{i_n,i_1}(H_{i_1}(r_1 + s_1) - H_{i_1}(r_1))\hat{H}_{i_n}(r_2)$$

as i_1, i_n, r_2, s_1, r_1 are arbitrary, divide the formula by $r_2 s_1$ and let r_2, s_1 go to zero leads to

$$\lambda_{i_1,i_n} H'_{i_n}(0)\hat{H}'_{i_1}(r_1) = \lambda_{i_n,i_1} H'_{i_1}(r_1)\hat{H}'_{i_n}(0),$$

finally note that $\hat{H}'_i = 1/H_i$, thus $\lambda_{i_1,i_n}(H^2_{i_n})'(0) = \lambda_{i_n,i_1}(H^2_{i_1})'(r_1)$. □

Lemma 3.4.5. *For all $i \sim j$, let $W_{i,j} = \lambda_{i,j}A_j/2 = \lambda_{j,i}A_i/2$, there exists constant φ_j depends only on j, such that $f_{i,j}(x) = W_{i,j}x + \varphi_j$.*

Proof. As $(H_j^2(s))' = A_j$, there exists B_j such that $H_j^2(s) = A_j s + B_j$, therefore

$$f_{i,j} \circ h_j^{-1}(s) = \lambda_{i,j} H_j(s) = \lambda_{i,j}\sqrt{A_j s + B_j}.$$

On the other hand, $(h_j^{-1})'(s) = \frac{1}{\sqrt{A_j s + B_j}}$, thus for some C_j,

$$h_j^{-1}(s) = \frac{2}{A_j}\sqrt{A_j s + B_j} + C_j.$$

$f_{i,j}(h_j^{-1}(s)) = f_{i,j}(\frac{2}{A_j}\sqrt{A_j s + B_j} + C_j) = \lambda_{i,j}\sqrt{A_j s + B_j}$, which leads to

$$f_{i,j}(x) = W_{i,j}x + \varphi_j,$$

where φ_j is some constant depends only on j. Applying the time change

$$\tilde{D}(s) = \sum_i \frac{l_i(s) - \varphi_i}{\varphi_i},$$

the resulting process will be of jump rate

$$W_{i,j}\varphi_i\varphi_j(1 + T_j(t))$$

where $T_j(t)$ is the local time for the time changed process $Z_t = X_{\tilde{D}^{-1}(t)}$. □

CHAPTER 4

SPEED OF VRJP ON GW TREES

(based on a joint work with X.Chen) [12]

Abstract

We give an alternative proof of the fact that the vertex reinforced jump process on Galton-Watson tree has a phase transition between recurrence and transience as a function of c, the initial local time, see [8]. Further, applying the techniques in [1], we show a phase transition between positive speed and null speed for the discrete time process associated in the transient regime.

4.1 Introduction and results

Suppose that $\mathcal{G} = (V, E)$ is a locally finite graph where V is the set of vertices and E is the collection of edges. Assign to each edge $e = \{u, v\} \in E$ a positive value $W_e = W_{u,v}$ as its weight, and assign to each vertex u a positive value ϕ_u as the initial local time. Define a continuous-time V valued process $(Y_t; t \geq 0)$ on $\mathcal{G}$ in the following way: At time 0 it starts at some vertex $v_0 \in V$; If $Y_t = v \in V$, then conditionally on $\{Y_s; 0 \leq s \leq t\}$, the process jumps to a neighbor u of v at rate $W_{v,u} L_u(t)$ where

$$L_u(t) := \phi_u + \int_0^t \mathbf{1}_{\{Y_s = u\}} ds. \tag{4.1}$$

The process $(Y_t)_{t\geq 0}$ is called vertex reinforced jump process (VRJP), first investigated in [18].

It has been proved in [18] that when $\mathcal{G} = \mathbb{Z}$, (Y_t) is recurrent. When $\mathcal{G} = \mathbb{Z}^d$ with $d \geq 2$, the complete description of its behavior has not been revealed even though lots of effort has been made, see e.g. [4, 8, 15, 18, 19, 44].

Here we are interested in the case when $\mathcal{G}$ is a supercritical Galton-Watson tree conditioned on its non-extinction, where acyclic property of trees largely reduces the difficulty to study this model. Indeed, in [15] it is shown that the VRJP on 3-regular tree has positive speed and satisfies a central limit theorem. Later, Basdevant and Singh [8] gave a precise description of the phase transition of recurrence/transience for VRJP on a supercritical Galton-Watson tree. In this paper, our main results, Theorem 4.1.2, describes the ballistic case of the VRJP when it is transient on a supercritical Galton-Watson tree without leaves. Our proof is based on the random walk in random environment (RWRE) representation result of Sabot and Tarrés [44], and the techniques on the studies of RWRE on a tree, especially a result of Aidekon [1] (see also e.g.[29, 30] for more on the studies of RWRE on trees).

Consider a rooted Galton-Watson tree T with offspring distribution $(q_k, k \geq 0)$ such that

$$b := \sum_{k\geq 0} kq_k > 1.$$

For some constant $c > 0$, we denote VRJP(c) the process (Y_t) on the Galton-Watson tree $T = (V, E)$ with $W_e \equiv 1, \forall e \in E$ and $\phi_x \equiv c, \forall x \in V$, starting from the root ρ. Hence the behaviors of this process depends on the graph $\mathcal{G}$ and on c. This definition is equivalent to VRJP with constant edge weight W and initial local time 1, up to a time change. We first recall the phase transition result obtained in [18]. Let A be an inverse Gaussian distribution of parameters $(1, c^2)$, i.e.

$$\mathbf{P}(A \in dx) = \mathbb{1}_{x\geq 0} \frac{c}{\sqrt{2\pi x^3}} \exp\left\{ -\frac{c^2(x-1)^2}{2x} \right\} dx, \tag{4.2}$$

The expectation w.r.t. $\mathbf{P}(dx)$ is denoted $\mathbf{E}$.

Theorem 4.1.1 (Basdevant & Singh). *Let $\mu(c) = \inf_{a\in\mathbb{R}} \mathbf{E}[A^a] = \mathbf{E}[\sqrt{A}]$, then the VRJP($c$) on a GW tree with offspring mean b is recurrent a.s. if and only if $b\mu(c) \leq 1$.*

Remark 4.1.1. *This phase transition was proved in [8] by considering the local times of VRJP. We will give another proof from the point of view of a random walk in random environment (RWRE), as a consequence of Theorem 4.2.1.*

When $b\mu(c) > 1$, a further question is to study the rate of escape of the process. Define the speed of the process (Y) by

$$v(Y) := \liminf_{t\to\infty} \frac{d(\rho, Y_t)}{t} = \lim_{t\to\infty} \frac{d(\rho, Y_t)}{t} \tag{4.3}$$

where d is the graph distance, and the last equality will be justified by Lemma 4.4.1. To study the speed, we use the RWRE point of view, heavily relying on a result of Sabot & Tarrès [45], in particular, the following fact:

Let (Y_t) be a VRJP on a finite graph $\mathcal{G} = (V, E)$ with edge weight (W) and initial local time (ϕ). If (Z_t) is defined by

$$Z_t := Y_{D^{-1}(t)} \text{ where } D(t) := \sum_{x\in V} (L_x(t)^2 - \phi_x^2), \tag{4.4}$$

then (Z_t) is a mixture of Markov jump processes (c.f. also [46]). Moreover, the mixing measure is explicit.

Applying this result to our VRJP(c) on a tree, denote $(\eta_n)_{n\geq 0}$ the discrete time process associated to (Z_t), it turns out that (η_n) is a random walk in random environment. In [1], for a random walk in random environment on a Galton-Watson tree, where the environment is site-wise independent and identically distributed, Aidekon gave a sharp and explicit criterion for the asymptotic speed to be positive. This result cannot apply directly to the time changed VRJP(c) on a tree, since the quenched transition probability depends also on the environment of the neighbors, see (4.7).

Aidekon's idea was to firstly seek for long branches on the GW tree, then compare the random walk to an auxiliary random walk on the half line, with the same type of environment. Thanks to the i.i.d. structure of the environment, he obtains sharp estimates for the one dimensional random walk, which allows him to come back to the tree without losing too much information. This also explains why the criterion depends on q_1, the probability that the GW tree generate one offspring.

Since our environment is also i.i.d., the same idea also work for the VRJP. Compare to [1], we mainly deal with the local dependences of the quenched probability transition. We believe that same type of criterion also holds for a larger type of random walk in random environment, with suitable conditions on the moments of the environment and locality of the transition probabilities.

Let us state our criterion, similar to (4.3), define

$$v(Z) = \liminf_{t\to\infty} \frac{d(\rho, Z_t)}{t},\ v(\eta) = \liminf_{n\to\infty} \frac{d(\rho, \eta_n)}{n}. \tag{4.5}$$

For the study of speed, we are only able to consider trees without leaves, hence we assume that $q_0 = 0$. In addition, we assume that

$$M := \sum_{k\geq 0} k^2 q_k < \infty.$$

For any $r \in \mathbb{R}$, let

$$\xi_r = \xi_r(c) := \mathbf{E}[A^{-r}].$$

It is clear that $\xi_r \in (0, \infty)$ for any r. In particular, $\mu(c) = \xi_{-1/2}(c)$. We see in the following theorem that the speed depends on the value of q_1.

Theorem 4.1.2. *Consider VRJP(c) on a supercritical GW tree such that $b\mu(c) > 1$, we have*

(1) $\lim_{t\to\infty} \frac{d(\rho,Z_t)}{t}$ *and* $\lim_{n\to\infty} \frac{d(\rho,\eta_n)}{n}$ *exist almost surely,*

(2) Assume $q_0 = 0$ and $M < \infty$. If $q_1\xi_{1/2} < 1$, then $v(\eta) > 0,\ v(Z) > 0$; if $q_1\xi_{1/2} > 1$, then $v(\eta) = v(Z) = 0$.

Corollary 4.1.1. *VRJP(c) $(Y_t)_{t\geq 0}$ on a supercritical GW tree such that $b\mu(c) > 1$, admits a speed $v(Y) \geq 0$ a.s. If in addition $q_0 = 0,\ M < \infty$ and $q_1\xi_{1/2} < 1$, then $v(Y) > 0$.*

Remark 4.1.2. *Our method cannot tackle the critical case $q_1\xi_{1/2} = 1$. Moreover, whether $q_1\xi_{1/2} > 1$ implies $v(Y) = 0$ remains unknown.*

The rest of this paper is organized as follows. In Section 4.2, we use a result of Sabot & Tarres [45] to recover the RWRE structure of VRJP. Section 3 is devoted to an alternative proof of Theorem 4.1.1, as an application of the RWRE point of view. Section 4 establishes the existence of the speed for the RWRE and Theorem 4.1.2. The proofs of some technical lemmas are in the last two sections.

4.2 RWRE on Galton-Watson tree

4.2.1 Mixture of Markov jump process by changing times

In this subsection, we consider a VRJP $(Y_t)_{t\geq 0}$ on a tree $T = (V, E)$ rooted at ρ, with edge weights (W) and initial local time (ϕ). If $x \neq \rho$, let $\overleftarrow{x}$ be the parent of x on the tree, the associated edge is denoted by $e_x = (x, \overleftarrow{x})$ with weight W_{e_x}.

Recall that the time changed version of VRJP (Z_t) defined in (4.4) is mixture of Markov jump processes with correlated mixing measure. The advantage of considering VRJP on trees is that, the random environment becomes independent.

Theorem 4.2.1. *Consider any tree $T = (V, E)$ rooted at ρ, endowed with edge weight $(W_e)_{e\in E}$ and initial local time $(\phi_x)_{x\in V}$. Let $(A_x, x \in V \setminus \{\rho\})$ be independent random variables such that*

$$\mathbf{P}(A_x \in da) = \mathbb{1}_{\mathbb{R}^+}(a)\phi_x \sqrt{\frac{W_{e_x}\phi_x\phi_{\overleftarrow{x}}}{2\pi a^3}} \exp(-W_{e_x}\phi_x\phi_{\overleftarrow{x}}\frac{(a-1)^2}{2a})da.$$

If X_t is a mixture of Markov jump processes starting from ρ, such that, conditionally on $(A_x, x \in V \setminus \{\rho\})$, X_t jumps from x to $\overleftarrow{x}$ at rate $\frac{1}{2}W_{e_x}\frac{\phi_{\overleftarrow{x}}}{\phi_x A_x}$ and from $\overleftarrow{x}$ to x at rate $\frac{1}{2}W_{e_x}\frac{\phi_x A_x}{\phi_{\overleftarrow{x}}}$. Then X_t and Z_t (defined in (4.4)) has the same distribution.

Proof. On a tree, VRJP observed at times when it stays on any finite sub-tree $T_f = (V_f, E_f)$ (also rooted at ρ) of T, behaves the same way as VRJP restricted to T_f; moreover, the restriction is independent of the VRJP outside T_f. Therefore, it is enough to prove the theorem on finite tree T_f. By Theorem 2 of [45] (with a slight modification of the initial local time), if we denote

$$l_x(t) = \int_0^t \mathbb{1}_{Z_s=x} ds,$$

then

$$U_x = \frac{1}{2}\lim_{t\to\infty}\left(\log\frac{l_x(t)+\phi_x^2}{l_\rho(t)+\phi_\rho^2} - \log\frac{\phi_x^2}{\phi_\rho^2}\right)$$

exists a.s. and $\{U_x, x \in V_f,\ U_\rho = 0\}$ has distribution (where $du = \prod_{x\neq\rho} du_x$)

$$dQ^{W,\phi}_{\rho,T_f}(u) = \frac{\prod_{x\neq\rho}\phi_x}{\sqrt{2\pi}^{|V_f|-1}} e^{-\sum_{x\in V_f} u_x - \sum_{\{x,y\}\in E_f}\frac{1}{2}W_{x,y}(e^{u_x-u_y}\phi_y^2 + e^{u_y-u_x}\phi_x^2 - 2\phi_x\phi_y)}\sqrt{\prod_{\{x,y\}\in E_f} W_{x,y}e^{u_x+u_y}}du.$$

Now, conditionally on (U_x), Z_t is a Markov process which jumps at rate (from x to z) $\frac{1}{2}W_{x,z}e^{U_z-U_x}$. For $e_x = (x, \overleftarrow{x}) \in T_f$, if we writes $y_{e_x} = (u_{\overleftarrow{x}} - \log\phi_{\overleftarrow{x}}) - (u_x - \log\phi_x)$, then (note that $u \mapsto y$ is a diffeomorphism and $dy = du$) the density of (u) also writes

$$dQ^{W,\phi}_{\rho,T_f}(u) = \prod_{e_x=\{x,\overleftarrow{x}\}\in E_f}\sqrt{\frac{W_{e_x}\phi_x\phi_{\overleftarrow{x}}}{2\pi}}\exp\left(\frac{1}{2}(y_{e_x} - W_{e_x}\phi_x\phi_{\overleftarrow{x}}(e^{y_{e_x}} + e^{-y_{e_x}} - 2))\right)dy.$$

Plugging $a_x = e^{-y_{e_x}}$ entails that a_x is Inverse Gaussian distributed with parameter $(1, W_{e_x}\phi_x\phi_{\overleftarrow{x}})$ and

$$dQ^{W,\phi}_{\rho,T_f}(a) = \prod_{x\in V_f\setminus\{\rho\}}\mathbb{1}_{a_x>0}\sqrt{\frac{W_{e_x}\phi_x\phi_{\overleftarrow{x}}}{2\pi a_x^3}}\exp(-W_{e_x}\phi_x\phi_{\overleftarrow{x}}\frac{(a_x-1)^2}{2a_x})da_x$$

Finally note that

$$\frac{1}{2}W_{x,z}e^{u_z-u_x} = \begin{cases}\frac{1}{2}W_{x,z}\frac{\phi_z}{\phi_x a_x} & \text{if } z = \overleftarrow{x}\\ \frac{1}{2}W_{x,z}\frac{\phi_z a_z}{\phi_x} & \text{if } \overleftarrow{z} = x.\end{cases}$$

□

For VRJP(c) on a GW tree, the theorem immediately implies:

Corollary 4.2.1. *On a GW tree $T = (V, E)$, the time changed VRJP(c) (Z_t) is a random walk in i.i.d. environment $(A_x, x \in V \setminus \{\rho\})$, where (A_x) are i.i.d. inverse Gaussian distributed with parameters $(1, c^2)$, and conditionally on the environment, the process jumps at rate*

$$\begin{cases}\frac{1}{2A_x} & \text{from } x \text{ to } \overleftarrow{x}\\ \frac{1}{2}A_x & \text{from } \overleftarrow{x} \text{ to } x.\end{cases} \tag{4.6}$$

4.2.2 RWRE on Galton Watson tree and notations

In the sequel, let $T = (V, E)$ be a Galton-Watson tree with offspring distribution $\{q_k; k \geq 0\}$. Recall that $(\eta_n)_{n\geq 0}$ denotes the discrete time process associated to (Z_t) (or (Y_t)), which is a random walk in random environment.

Note that there are two randomnesses of the environment. First, we sample a GW tree, T, whose law is denoted by $GW(dT)$. Then, given the tree T (rooted at ρ), we define $\omega = \{A_x, x \in V \setminus \{\rho\}\}$ as in Corollary 4.2.1, whose law is defined under $\prod_{x\in T\setminus\{\rho\}} \mathbf{P}(dA_x)$, which we denote abusively $\mathbf{P}(d\omega)$. Finally, given (w, T), the Markov jump process $(Z_t; t \geq 0)$ is defined by its jump rate in (4.6).

For convenience, we artificially add a vertex $\overleftarrow{\rho}$ to T, designing the parent of the root. Let A_ρ be another copy of A, independent of all others. Now, let $\omega = (A_x, x \in V)$ be the enlarged environment. Given (ω, T), define the new Markov chain η, which is a random walk on $V \cup \{\overleftarrow{\rho}\}$, with transition probabilities

$$\begin{cases} p(x, \overleftarrow{x}) = \frac{1}{1+A_x \sum_{y:\, \overleftarrow{y}=x} A_y} \\ p(x, z) = \frac{A_x A_z}{1+A_x \sum_{y:\, \overleftarrow{y}=x} A_y} \quad \text{where } \overleftarrow{z} = x \in V \\ p(\overleftarrow{\rho}, \rho) = 1 \end{cases} \tag{4.7}$$

This modification will not change the recurrence/transience behavior of the RWRE η nor its speed in the transient regime.

Let us now introduce the notation of quenched and annealed probabilities. Given the environment (ω, T), let $P_x^{\omega,T}$ denote the quenched probability of the random walk η with $\eta_0 = x \in V$ a.s. Denote by $\mathbb{P}_x^T$, $\mathbf{Q}$, $\mathbb{P}_\rho$ the mesures:

$$\mathbb{P}_x^T(\cdot) := \int P_x^{\omega,T}(\cdot)\mathbf{P}(d\omega),$$

$$\mathbf{Q}(\cdot) := \int 1_{\{\cdot\}}\mathbf{P}(d\omega)GW(dT)$$

$$\mathbb{P}_\rho(\cdot) := \int \mathbb{P}_\rho^T(\cdot)GW(dT).$$

For brevity, write $P^{\omega,T}$, $\mathbb{P}^T$ and $\mathbb{P}$ for $P_\rho^{\omega,T}$, $\mathbb{P}_\rho^T$ and $\mathbb{P}_\rho$, and the associated expectations are denoted $E_x^{\omega,T}$, $\mathbb{E}_x^T$ and $\mathbb{E}$. Notice that $\mathbb{P}$ is the annealed law of η. Finally, the expectation corresponds to $\mathbf{Q}$ is denoted $\mathbb{E}_{\mathbf{Q}}$.

On the tree T rooted at ρ, for any vertex x, let $|x| = d(\rho, x)$ be the generation of x and denote by $[\![\rho,\ x]\!]$ the unique shortest path from x to the root ρ, and x_i (for $0 \leq i \leq |x|$) the vertices on $[\![\rho,\ x]\!]$ such that $|x_i| = i$. Thus, $x_0 = \rho$ and $x_{|x|} = x$. In words, x_i (for $i < |x|$) is the ancestor of x at generation i. Also denote $]\!]\rho,\ x]\!] := [\![\rho,\ x]\!]\setminus\{\rho\}$ and $]\!]\rho,\ x[\![:= [\![\rho,\ x]\!]\setminus\{\rho, x\}$.

4.3 An alternative proof of phase transition

The ideas follow from Lyons and Pemantle [41], by means of random electrical network.

Proof of Theorem 4.1.1. Recall that the environment ω is given by i.i.d. random variables A_x, $x \in T$, with inverse Gaussian distribution $IG(1, c^2)$. The RWRE is equivalent to an electrical network with random conductances:

$$C_{e_x} := C(x, \overleftarrow{x}) = (\prod_{u\in]\!]\rho,x[\![} A_u)^2 A_x, \forall x \in V \setminus \{\rho\}.$$

We omit the proof of the transient case which is quite similar to that in Lyons and Pemantle [41], however, we will detail the recurrence case. That is, we will show that if $b\mu(c) \leq 1$, then the RWRE is recurrent a.s.

First consider the case $b\mu(c) < 1$, note that

$$\begin{aligned}\mathbb{E}_{\mathbf{Q}}\Big[\sum_{n\geq 1}\sum_{|x|=n} C_{e_x}^{1/4}\Big] &= \sum_{n\geq 1}\int\Big(\int\sum_{|x|=n} C_{e_x}^{1/4}\mathbf{P}(d\omega)\Big)GW(dT)\\ &= \sum_{n\geq 1}\int\sum_{|x|=n}\mathbf{E}[A^{1/2}]^{n-1}\mathbf{E}[A^{1/4}]GW(dT)\\ &= \sum_{n\geq 1} b^n\mathbf{E}[A^{1/2}]^{n-1}\mathbf{E}[A^{1/4}].\end{aligned}$$

Because $\mu(c) = \mathbf{E}[A^{1/2}] < 1/b$, we have, for some constants $c_1, c_2 \in \mathbb{R}^+$

$$\mathbb{E}_{\mathbf{Q}}\Big[\sum_{n\geq 1}\sum_{|x|=n} C_{e_x}^{1/4}\Big] \leq c_1\sum_{n\geq 0}(b\mu(c))^n \leq c_2 < \infty,$$

which implies that

$$\sum_{n\geq 1}\sum_{|x|=n} C_{e_x}^{1/4} < \infty,\ \mathbf{Q}\text{-a.s.}$$

As a result, there exists a stationary probability a.s., moreover η is positive recurrent.

Turning to the case $b\mu(c) = 1$, let $\Pi_n := \{e_x : |x| = n\}$ be a sequence of cutsets. Observe that

$$W_n := \sum_{|x|=n}\prod_{u\in]\!]\rho,x]\!]} A_u^{1/2} = \sum_{|x|=n} C_{e_x}^{1/4}A_x^{1/4}.$$

is a martingale with respect to its natural filtration. By Biggin's theorem [2, 32], it converges a.s. to zero. We are going to show that $\mathbf{Q}$-a.s.,

$$\liminf_{n\to\infty}\sum_{|x|=n} C_{e_x}^{1/4} = 0, \tag{4.8}$$

which implies that $\inf_{\Pi:\ \text{cutset}}\sum_{e_x\in\Pi} C_{e_x} = 0$. By the trivial half of the max-flow min-cut theorem, the corresponding network admits no flow a.s. Hence, no electrical current flows. This implies that the random walk is a.s. recurrent.

One observes that

$$\begin{aligned}\sum_{|x|=n} C_{e_x}^{1/4} &= \sum_{|x|=n}\prod_{u\in]\!]\rho,x[\![} A_u^{1/2}A_x^{1/4}1_{\{A_x\geq 1\}} + \sum_{|x|=n}\prod_{u\in]\!]\rho,x[\![} A_u^{1/2}A_x^{1/4}1_{\{A_x< 1\}}\\ &\leq \sum_{|x|=n}\prod_{u\in]\!]\rho,x]\!]} A_u^{1/2}A_x^{-1/4}1_{\{A_x\geq 1\}} + \sum_{|y|=n-1}\prod_{u\in]\!]\rho,y]\!]} A_u^{1/2}\sum_{x:\overleftarrow{x}=y} A_x^{1/4}1_{\{A_x<1\}}\\ &\leq W_n + \sum_{|y|=n-1}\prod_{u\in]\!]\rho,y]\!]} A_u^{1/2}\nu_y,\end{aligned}$$

where ν_y denotes the number of children of y. Letting n go to infinity yields that

$$0 \leq \liminf_{n\to\infty}\sum_{|x|=n} C_{e_x}^{1/4} \leq \liminf_{n\to\infty}\sum_{|y|=n-1}\prod_{u\in]\!]\rho,y]\!]} A_u^{1/2}\nu_y.$$

For any $K \geq 1$, separating the sum over vertices y according to $\{\nu_y < K\}$ or $\{\nu_y \geq K\}$, the last term is bounded by

$$\begin{aligned}&\lim_{n\to\infty} KW_{n-1} + \liminf_{n\to\infty}\sum_{|y|=n-1}\prod_{u\in]\!]\rho,y]\!]} A_u^{1/2}\nu_y 1_{\{\nu_y\geq K\}}\\ &= \liminf_{n\to\infty}\sum_{|y|=n-1}\prod_{u\in]\!]\rho,y]\!]} A_u^{1/2}\nu_y 1_{\{\nu_y\geq K\}}.\end{aligned}$$

It follows then from the Fatou's lemma that

$$\begin{aligned}
&\mathbb{E}_{\mathbf{Q}}\Big(\liminf_{n\to\infty}\sum_{|y|=n-1}\prod_{u\in\rrbracket\rho,y\rrbracket}A_u^{1/2}\nu_y 1_{\{\nu_y\geq K\}}\Big)\\
&\leq\liminf_{n\to\infty}\mathbb{E}_{\mathbf{Q}}\Big(\sum_{|y|=n-1}\prod_{u\in\rrbracket\rho,y\rrbracket}A_u^{1/2}\nu_y 1_{\{\nu_y\geq K\}}\Big)=\mathbb{E}_{\mathbf{Q}}[\nu_\rho,\nu_\rho\geq K],
\end{aligned}$$

since for all $|y|=n-1$, ν_y is independent of $\prod_{u\in\rrbracket\rho,y\rrbracket}A_u^{1/2}$ and $\mathbb{E}_{\mathbf{Q}}\Big(\sum_{|y|=n-1}\prod_{u\in\rrbracket\rho,y\rrbracket}A_u^{1/2}\Big)=1$. Consequently, for any $K\geq 1$,

$$\mathbb{E}_{\mathbf{Q}}\Big[\liminf_{n\to\infty}\sum_{|x|=n}C_{e_x}^{1/4}\Big]\leq\mathbb{E}_{\mathbf{Q}}[\nu_\rho,\nu_\rho\geq K].$$

As $b=\mathbb{E}_{\mathbf{Q}}[\nu_y]<\infty$, letting $K\to\infty$ gives

$$\mathbb{E}_{\mathbf{Q}}\Big[\liminf_{n\to\infty}\sum_{|x|=n}C_{e_x}^{1/4}\Big]=0.$$

This implies (4.8). □

4.4 Speed when transient

We are going to study the positivity of $v(Z)$ and $v(\eta)$, using the fact that the processes (Z_t) and (η_n) are mixture of Markov processes. The process (Y_t) escapes faster than (Z_t), in particular, when $v(Z)>0$, we have $v(Y)>0$. But we are not sure whether $v(Z)=0$ implies $v(Y)=0$.

4.4.1 Regeneration structure

In this section, we show that, when the process (η_n) (or (Z_t)) is transient, its path can be cut into independent pieces, using the notion of regeneration time. As a consequence, the speed $v(\eta)$, $v(Z)$ exists a.s.

On a tree, when a random walk traverses an edge for the first and last time simultaneously, we say it regenerates since it will now remain in a previously unexplored sub-tree. For any vertex x, let $\mathcal{D}(x)=\inf\{k\geq 1,\ \eta_{k-1}=x,\eta_k=\overleftarrow{x}\}$, write $\tau_n=\inf\{k\geq 0,\ |\eta_k|=n\}$ and define the regeneration time recursively by

$$\begin{cases}\Gamma_0=0\\ \Gamma_n=\Gamma_n(\eta)=\inf\{k>\Gamma_{n-1};\ d(\eta_k)\geq 3,\mathcal{D}(\eta_k)=\infty,\tau_{|\eta_k|}=k\}.\end{cases}$$

where $d(x)$ is the degree of the vertex x.

Lemma 4.4.1. *Let $S(\cdot)=\mathbb{P}(\cdot|d(\rho)\geq 3,\ \mathcal{D}(\rho)=\infty)$, if η is transient, then*

i) For any $n\geq 1$, $\Gamma_n<\infty$ $\mathbb{P}$-a.s.

ii) Under $\mathbb{P}$, $(\Gamma_{n+1}-\Gamma_n,|\eta_{\Gamma_{n+1}}|-|\eta_{\Gamma_n}|,A_{\Gamma_{n+1}})_{n\geq 1}$ are independent and distributed as $(\Gamma_1,|\eta_{\Gamma_1}|,A_{\Gamma_1})$ under S.

iii) $E_S(|\eta_{\Gamma_1}|)<\infty$.

We feel free to omit the proof because it is analogue to 'Fact' in [1] p.10. In addition, Lemma 4.4.1 also holds without assuming $d(\eta_k) \geq 3$ in the definition of Γ_n, but we will need this assumption later in the proof of Lemma 4.4.7.

By strong law of large numbers, one immediately sees that there exist two constants $c_4 \geq c_3 \geq 1$ such that $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty} \frac{|\eta_{\Gamma_n}|}{n} = c_3 \in [1, \infty), \quad \lim_{n\to\infty} \frac{\Gamma_n}{n} = c_4 \in [c_3, \infty].$$

In addition, for any $n \geq 1$, there exists a unique $u(n) \in \mathbb{N}$ such that

$$\Gamma_{u(n)} \leq n < \Gamma_{u(n)+1}$$

and $|\eta_{\Gamma_{u(n)}}| \leq |\eta_n| < |\eta_{\Gamma_{u(n)+1}}|$. Letting n go to infinity, (in particular $u(n) \to \infty$) in

$$\frac{|\eta_{\Gamma_{u(n)}}|}{\Gamma_{u(n)+1}} \leq \frac{|\eta_n|}{n} < \frac{|\eta_{\Gamma_{u(n)+1}}|}{\Gamma_{u(n)}} = \frac{|\eta_{\Gamma_{u(n)+1}}|}{u(n)} \frac{u(n)}{\Gamma_{u(n)}}.$$

We have $\mathbb{P}$-a.s.

$$\frac{|\eta_n|}{n} \to v(\eta) := \frac{c_3}{c_4} \in [0, 1].$$

For Z_t, the same arguments can be applied. As a consequence of the i.i.d. decomposition, $v(Z) = \lim_{t\to\infty} \frac{|Z_t|}{t}$ exists a.s. The existence of $v(Y) = \lim_{t\to\infty} \frac{|Y_t|}{t}$ can be justified by performing the time change $D(t)$ between consecutive regenerative epochs.

4.4.2 The auxiliary one dimensional process

The RWRE can also be defined on the deterministic graph $\mathbb{H} = \{-1, 0, 1, \dots\}$, on which many quantities are viable by explicit computations. The strategy is to compare the random walk on a tree to the random walk on the half line, in the forth coming sections we will explain how these comparisons will be done. In this section we list some properties of the one dimensional random walk, their proofs can be found in Section 4.5.

Let $\tilde{\eta}_n$ be the random walk on the half line $\mathbb{H} = \{-1, 0, 1, \dots\}$ in the random environment $\omega = (A_k, k \geq 0)$ which are i.i.d. copies of A under $\mathbf{P}$, with transition probability according to (4.7); that is,

$$\begin{cases} p(i, i+1) = \frac{A_{i+1}}{1/A_i + A_{i+1}} & i \geq 0 \\ p(i, i-1) = \frac{1/A_i}{1/A_i + A_{i+1}} & i \geq 0 \\ p(-1, 0) = 1 \end{cases}$$

Similarly we denote $\tilde{P}_i^{\omega}, \tilde{\mathbb{P}}_i, \tilde{E}_i^{\omega}, \tilde{\mathbb{E}}_i$ respectively the quenched and annealed probability/expectation for such process starting from i, and for any $n \in \mathbb{H}$, define the following stopping times

$$\tilde{\tau}_n = \inf\{k \geq 0,\ \tilde{\eta}_k = n\}, \quad \tilde{\tau}_n^* = \inf\{k \geq 1,\ \tilde{\eta}_k = n\}.$$

Let $F_1, F_2 > 0$ be two expressions which can depend on any variable, and in particular on n. If there exists $f : \mathbb{N} \to \mathbb{R}^+$ with $\lim_{n\to\infty} \frac{1}{n} \log f(n) = 0$ such that $F_1 f(n) \geq F_2$, then we denote $F_1 \gtrsim F_2$ (F_1 greater than F_2 up to polynomial constant). If $F_1 \gtrsim F_2$ and $F_1 \lesssim F_2$, then we write $F_1 \simeq F_2$.

Recall that A is Inverse Gaussian distributed with parameter $(1, c^2)$, define the rate function associated to $\log A$ by

$$I(x) = \sup_{t\in\mathbb{R}}\{tx - \log \mathbf{E}(A^t)\}, \tag{4.9}$$

also define

$$t^* = \sup\{t \in \mathbb{R},\ \mathbf{E}(A^t) q_1 \leq 1\}. \tag{4.10}$$

Lemma 4.4.2. *For any $z > 0$ and $0 < z_1 < 1$, we have, for any $0 < a < 1$*

$$\tilde{\mathbb{P}}_0(\tilde{\tau}_n \wedge \tilde{\tau}_{-1} > m | A_0 \in [a, \frac{1}{a}]) \gtrsim \exp\{-n\left(z_1 I(\frac{z}{2z_1}) + (1-z_1) I(\frac{-z}{2(1-z_1)})\right)\}$$

where $m \in \mathbb{N}$ is such that $n = \lfloor \frac{\log m}{z} \rfloor$.

Lemma 4.4.3. *Denote*

$$L' = \sup_{z>0,\, 0<z_1<1} \{\frac{\log q_1}{z} - \frac{z_1}{z} I(\frac{z}{2z_1}) - \frac{1-z_1}{z} I(\frac{-z}{2(1-z_1)})\},$$

we have $L' = -t^* + \frac{1}{2}$.

Lemma 4.4.4. *Define, for $i \in \mathbb{H}$ and any stopping time τ, $\tilde{G}^\tau(i,i) = \tilde{E}_i^\omega(\sum_{k=0}^\tau \mathbb{1}_{\tilde{\eta}_k = i})$. Let $0 \le Y_1 < Y_2 < y < Y_3$ be points on the half line, we have, for any $0 \le \lambda \le 1$,*

$$\widetilde{P}_{Y_1}^\omega(\tilde{\tau}_y < \tilde{\tau}_{Y_1-1}) \widetilde{G}^{\tilde{\tau}_{Y_1} \wedge \tilde{\tau}_{Y_3}}(y,y) \le \widetilde{E}_{Y_1}^\omega[\tilde{\tau}_{Y_1-1} \wedge \tilde{\tau}_{Y_3}]. \tag{4.11}$$

$$\widetilde{E}_{Y_1}^\omega[\tilde{\tau}_{Y_1-1} \wedge \tilde{\tau}_{Y_3}]^\lambda \le S_{\lambda, [\![Y_1, Y_2]\!]}\left(1 + A_{Y_2+1}^\lambda\left(1 + \widetilde{E}_{Y_2+1}^\omega[\tilde{\tau}_{Y_2} \wedge \tilde{\tau}_{Y_3}]^\lambda\right)\right). \tag{4.12}$$

where

$$S_{\lambda, [\![Y_1, Y_2]\!]} := 1 + 2A_{Y_1}^\lambda \sum_{Y_1 < z \le Y_2} \prod_{Y_1 < u < z} A_u^{2\lambda} A_z^\lambda + A_{Y_1}^\lambda \prod_{Y_1 < u \le Y_2} A_u^{2\lambda}.$$

Lemma 4.4.5. *If $0 \le \lambda < (t^* - \frac{1}{2}) \wedge 1$, then there exists sufficiently small $\delta > 0$ such that for all $n_1 > 0$*

$$\mathbf{E}\left((1 + \frac{1}{A_{n_1}^\lambda})(1 + \frac{1}{A_n}) A_0 \widetilde{E}_0^\omega[\tilde{\tau}_{-1} \wedge \tilde{\tau}_n]^\lambda\right) \lesssim (q_1 + \delta)^{-n}.$$

4.4.3 Null speed case

In this section we prove (2) of Theorem 4.1.2.

Proposition 4.4.1. *Recall the definition of t^* in (4.10), if $q_1 \mathbf{E}(A^{-1/2}) > 1$, then $1 < t^* < \frac{3}{2}$ and*

$$\limsup_n \frac{\log |\eta_n|}{\log n} \le t^* - \frac{1}{2}.$$

In particular, if $q_1 \mathbf{E}(A^{-1/2}) > 1$, then $\mathbb{P}$-a.s., $v(\eta) = 0$; in fact,

$$|\eta_n| = n^{(t^* - 1/2) + o(1)} = o(n), \ n \to \infty.$$

Remark 4.4.1. *Similar arguments can be carried out for the continuous time process (Z_t), i.e. if $q_1 \mathbf{E}(A^{-1/2}) > 1$, then*

$$\limsup_t \frac{\log |Z_t|}{\log t} \le t^* - \frac{1}{2}. \tag{4.13}$$

Let us state an estimate on the tail distribution of the regeneration time Γ_1 under $S(\cdot)$:

Lemma 4.4.6.

$$S(\Gamma_1 > n) \gtrsim n^{-t^* + \frac{1}{2}} \tag{4.14}$$

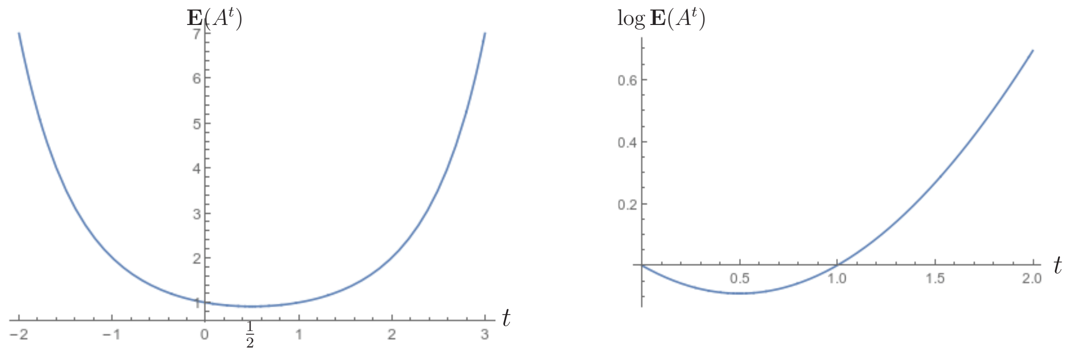

Figure 4.1: The function $t \mapsto \mathbf{E}(A^t)$ and $t \mapsto \log \mathbf{E}(A^t)$ for $c = 1$.

With the help of the above lemma, we prove Proposition 4.4.1.

Proof of Proposition 4.4.1. Note that $t \mapsto \mathbf{E}(A^t)$ is a convex function, and it is symmetric w.r.t. the line $t = \frac{1}{2}$, where it takes the minimum, in particular $\mathbf{E}(A^{-1/2}) = \mathbf{E}(A^{3/2})$. As we have assumed that $q_1\mathbf{E}(A^{-1/2}) > 1$, it follows that $t^* < \frac{3}{2}$. On the other hand, since $\mathbf{E}(A) = 1$, obviously $t^* > 1$. For any $\lambda \in (t^* - 1/2, 1)$, by Lemma 4.4.6, there exists $\varepsilon > 0$ such that

$$\begin{aligned}\mathbb{P}(\max_{2\le k\le n}(\Gamma_k - \Gamma_{k-1}) \le n^{1/\lambda}) &= S(\Gamma_1 \le n^{1/\lambda})^{n-1} \\ &\le (1 - n^{-1+\varepsilon})^{n-1} \lesssim \exp(-n^{\varepsilon}).\end{aligned}$$

Therefore,

$$\sum_{n\ge 2} \mathbb{P}(\max_{2\le k\le n}(\Gamma_k - \Gamma_{k-1}) \le n^{1/\lambda}) < \infty.$$

By Borel-Cantelli lemma, $\mathbb{P}$-a.s., for all n large enough,

$$\Gamma_n \ge \max_{2\le k\le n}(\Gamma_k - \Gamma_{k-1}) \ge n^{1/\lambda}.$$

It follows that $\mathbb{P}$-a.s., $\liminf_n \frac{\log \Gamma_n}{\log n} \ge \frac{1}{\lambda}$. As $\liminf_n \frac{\log \tau_n}{\log n} \ge \liminf_n \frac{\log \Gamma_n}{\log n}$ (see (3.1) in [1]), we have

$$\limsup_n \frac{\log |\eta_n|}{\log n} \le \lambda \xrightarrow[\text{decreasing}]{} t^* - \frac{1}{2} < 1, \mathbb{P}\text{-a.s.}$$

□

In fact, when q_1 is large, it is more likely that there will be some long branch constituting vertices of degree two on the GW tree, especially starting from the root. These branches will slow down the process and entail zero velocity. The following lemma gives a comparison between the tail distribution of the regeneration time Γ_1 and the probability that the process wanders on these branches (which is a one dimensional random walk in random environment, that is, $(\tilde{\eta}_n)$).

Lemma 4.4.7. *For any $m \ge 1$, $0 < a < 1$, we have*

$$S(\Gamma_1 > m) \ge c_5 \sum_{n=1}^{\infty} q_1^n \tilde{\mathbb{P}}_0(\tilde{\tau}_{-1} \wedge \tilde{\tau}_n > m | A_0 \in [a, \frac{1}{a}]).$$

Now we prove Lemma 4.4.6 with the help of Lemma 4.4.7 and some results on the one dimensional RWRE.

Proof of Lemma 4.4.6. By Lemma 4.4.2, one sees that for $z > 0$, $0 < z_1 < 1$ and m such that $n = \lfloor \frac{\log m}{z} \rfloor$,

$$\tilde{\mathbb{P}}_0(\tilde{\tau}_n \wedge \tilde{\tau}_{-1} > m | A_0 \in [a, \frac{1}{a}]) \gtrsim \exp(-n \left(z_1 I(\frac{z}{2z_1}) + (1-z_1) I(\frac{-z}{2(1-z_1)}) \right))$$

where we recall that $I(x) = \sup_{t\in\mathbb{R}}\{tx - \log \mathbf{E}(A^t)\}$. For large m, by Lemma 4.4.7, then Lemma 4.4.2,

$$\begin{aligned} S(\Gamma_1 > m) &\geq c_5 \max_{n: n = \lfloor \frac{\log m}{z} \rfloor} q_1^n \tilde{\mathbb{P}}_0(\tilde{\tau}_n \wedge \tilde{\tau}_{-1} > m | A_0 \in [a, \frac{1}{a}]) \\ &\gtrsim \max_{n: n = \lfloor \frac{\log m}{z} \rfloor} q_1^n \exp(-n \left(z_1 I(\frac{z}{2z_1}) + (1-z_1) I(\frac{-z}{2(1-z_1)}) \right)) \\ &\gtrsim \sup_{z>0, z_1 \in (0,1)} \exp\{-\frac{\log m}{z} \left(z_1 I(\frac{z}{2z_1}) + (1-z_1) I(\frac{-z}{2(1-z_1)}) - \log q_1 \right)\}. \end{aligned}$$

It follows from Lemma 4.4.3 that

$$S(\Gamma_1 > m) \gtrsim m^{-t^*+1/2}.$$

□

It remains to prove the comparison Lemma 4.4.7. We define, for $x \neq \overleftarrow{\rho}$,

$$\tau_x = \inf\{n \geq 0;\ \eta_n = x\},\ \tau_x^* = \inf\{n > 0;\ \eta_n = x\},\ \beta(x) = P_x^{\omega,T}(T_{\overleftarrow{x}} = \infty)$$

Note that for any $x \in T$, $\beta(x)$ depends only on the sub-tree T_x rooted at x and the environment $\{A_y(\omega); y \in T_x\}$, let us denote β a generic r.v. distributed as $\beta(\rho)$, by transient assumption, $\beta > 0$ a.s. and $\mathbb{E}(\beta) > 0$.

Moreover, by Markov property,

$$\begin{aligned} \beta(x) &= \sum_{y:\, \overleftarrow{y}=x} p(x,y)[P_y^{\omega,T}(\tau_x = \infty) + P_y^{\omega,T}(\tau_x < \infty)\beta(x)] \\ &= \sum_{y:\, \overleftarrow{y}=x} p(x,y)[\beta(y) + (1-\beta(y))\beta(x)]. \end{aligned}$$

Note that $\beta(x) > 0$, $\mathbb{P}$-a.s. hence,

$$\frac{1}{\beta(x)} = 1 + \frac{1}{A_x \sum_{y:\, \overleftarrow{y}=x} A_y \beta(y)}. \tag{4.15}$$

In particular, $\beta(x)$ is increasing as a function of A_x.

Proof of Lemma 4.4.7. For any vertex x, let $h(x)$ be the first descendant of x such that $d(h(x)) \geq 3$. Let $k_0 = \inf\{k \geq 2 :\ q_k > 0\}$. According to the definition of Γ_1, one observes that when $\eta_1 \neq \overleftarrow{\rho}$,

$$\Gamma_1 \geq \tau_\rho^* \wedge \tau_{h(X_1)}.$$

In fact, we are going to consider the following events

$$\begin{aligned} E_0 &= \{d(\rho) = k_0 + 1,\ A_\rho \geq a,\ A_{\rho_i} \in [a, \frac{1}{a}], \forall 1 \leq i \leq k_0\} \text{ where } \rho_i \text{ are children of } \rho, \\ E_1 &= E_0 \cap \{\eta_1 \neq \overleftarrow{\rho}, m < \tau_\rho^* < \tau_{h(\eta_1)}, \eta_{\tau_\rho^*+1} \notin \{\overleftarrow{\rho}, \eta_1\}\} \cap \{\eta_n \neq \rho; \forall n \geq \tau_\rho^* + 1\}, \\ E_2 &= E_0 \cap \{\eta_1 \neq \overleftarrow{\rho}, m < \tau_{h(\eta_1)} < \tau_\rho^*\} \cap \{\eta_n \neq \overleftarrow{h(\eta_1)}, \forall n \geq \tau_{h(\eta_1)} + 1\}. \end{aligned}$$

As $\Gamma_1 \geq \tau^*_\rho \wedge \tau_{h(\eta_1)}$, we have $E_1 \cup E_2 \subset E_0 \cap \{D(\rho) = \infty, \Gamma_1 > m\}$ and $E_1 \cap E_2 = \emptyset$. So,

$$\mathbb{P}(E_0 \cap \{D(\rho) = \infty, \Gamma_1 > m\}) \geq \mathbb{P}(E_1) + \mathbb{P}(E_2).$$

For E_1, by strong Markov property at τ^*_ρ and weak Markov property at time 1,

$$\begin{aligned} P^{\omega,T}_\rho(E_1) &= \mathbb{1}_{E_0} P^{\omega,T}_\rho(\{\eta_1 \neq \overleftarrow{\rho}, m < \tau^*_\rho < \tau_{h(\eta_1)}, \eta_{\tau^*_\rho+1} \notin \{\overleftarrow{\rho}, \eta_1\}\} \cap \{\eta_n \neq \rho; \forall n \geq \tau^*_\rho + 1\}) \\ &= \mathbb{1}_{E_0} \sum_{i=1}^{k_0} p(\rho, \rho_i) P^{\omega,T}_{\rho_i}(m - 1 < \tau_\rho < \tau_{h(\rho_i)}) \sum_{j \neq i} p(\rho, \rho_j)\beta(\rho_j). \end{aligned}$$

Given E_0, $p(\rho, \rho_i) \geq \frac{a^2}{k_0+1} =: c_6$. So,

$$P^{\omega,T}_\rho(E_1) \geq c_6 \mathbb{1}_{E_0} \sum_{i=1}^{k_0} P^{\omega,T}_{\rho_i}(m - 1 < \tau_\rho < \tau_{h(\rho_i)}) \sum_{j \neq i} p(\rho, \rho_j)\beta(\rho_j),$$

Conditionally on $\{d(\rho), A_\rho, A_{\rho_i}, 1 \leq i \leq d(\rho) - 1\}$, the independence of the environment implies that

$$\begin{aligned} &\mathbb{P}\Big(E_1 \Big| d(\rho), A_\rho, A_{\rho_i}, 1 \leq i \leq d(\rho) - 1\Big) \\ &\qquad\qquad \geq c_6 \mathbb{1}_{E_0} \sum_{i=1}^{k_0} \mathbb{P}_{\rho_i}(m - 1 < \tau_\rho < \tau_{h(\rho_i)}) \sum_{j \neq i} p(\rho, \rho_j) \mathbf{E_Q}[\beta(\rho_j)|A_{\rho_j}], \end{aligned}$$

where, for each $j \neq i$, $p(\rho, \rho_j)$ and $\mathbf{E_Q}[\beta(\rho_j)|A_{\rho_j}]$ are increasing functions of A_{ρ_j}. By FKG inequality,

$$\begin{aligned} \mathbb{P}(E_1) &\geq c_6 \mathbb{E}\Big(\mathbb{1}_{E_0} \sum_{i=1}^{k_0} P^{\omega,T}_{\rho_i}(m - 1 < \tau_\rho < \tau_{h(\rho_i)}) \sum_{j \neq i} p(\rho, \rho_j)\Big) \times \mathbb{E}(\beta(\rho)) \\ &\geq c_7 \mathbb{E}\Big(\mathbb{1}_{E_0} \sum_{i=1}^{k_0} P^{\omega,T}_{\rho_i}(m - 1 < \tau_\rho < \tau_{h(\rho_i)})\Big), \end{aligned} \tag{4.16}$$

since $\mathbb{E}(\beta(\rho)) > 0$ and on E_0, $\sum_{j \neq i} p(\rho, \rho_j) \geq \frac{a^2(k_0-1)}{1+k_0} > 0$. Similarly for E_2, by Markov property,

$$\begin{aligned} P^{\omega,T}_\rho(E_2) &= \mathbb{1}_{E_0} P^{\omega,T}_\rho(\{\eta_1 \neq \overleftarrow{\rho}, m < \tau_{h(\eta_1)} < \tau^*_\rho\} \cap \{\eta_n \neq \overleftarrow{h(\eta_1)}; \forall n \geq \tau_{h(\eta_1)} + 1\}) \\ &= \mathbb{1}_{E_0} \sum_{i=1}^{k_0} p(\rho, \rho_i) P^{\omega,T}_{\rho_i}(m - 1 < \tau_{h(\rho_i)} < \tau_\rho)\beta(h(\rho_i)) \\ &\geq c_6 \mathbb{1}_{E_0} \sum_{i=1}^{k_0} P^{\omega,T}_{\rho_i}(m - 1 < \tau_{h(\rho_i)} < \tau_\rho)\beta(h(\rho_i)). \end{aligned}$$

Again $P^{\omega,T}_{\rho_i}(m - 1 < \tau_{h(\rho_i)} < \tau_\rho)$ and $\beta(h(\rho_j))$ are both increasing on $A_{h(\rho_i)}$. FKG inequality entails

$$\begin{aligned} \mathbb{P}(E_2) &\geq c_6 \mathbb{E}\Big(\mathbb{1}_{E_0} \sum_{i=1}^{k_0} P^{\omega,T}_{\rho_i}(m - 1 < \tau_{h(\rho_i)} < \tau_\rho)\Big) \times \mathbb{E}(\beta(\rho)) \\ &= c_8 \mathbb{E}(\mathbb{1}_{E_0} \sum_{i=1}^{k_0} P^{\omega,T}_{\rho_i}(m - 1 < \tau_{h(\rho_i)} < \tau_\rho)), \end{aligned} \tag{4.17}$$

with $c_8 := c_6\mathbb{E}(\beta(\rho)) > 0$. Combining (4.16) with (4.17) yields that

$$\begin{aligned}\mathbb{P}(E_1) + \mathbb{P}(E_2) &\geq c_9\mathbb{E}\Big(\mathbb{1}_{E_0}\sum_{i=1}^{k_0} P^{\omega,T}_{\rho_i}(\tau_\rho \wedge \tau_{h(\rho_i)} > m-1)\Big)\\ &\geq c_9 K_0 \mathbf{Q}(E_0)\mathbb{P}\Big(\tau_{\overleftarrow{\rho}} \wedge \tau_{h(\rho)} > m-1 | A_\rho \in [a, \frac{1}{a}]\Big)\\ &\geq c_{10}\mathbb{P}\Big(\tau_{\overleftarrow{\rho}} \wedge \tau_{h(\rho)} > m-1 | A_\rho \in [a, \frac{1}{a}]\Big). \end{aligned} \tag{4.18}$$

Let us go back to $S(\Gamma_1 > m)$. As $\mathbb{P}(d(\rho) \geq 3, D(\rho) = \infty) > 0$, recall that

$$\begin{aligned}S(\Gamma_1 > m) &= \mathbb{P}(\Gamma_1 > m | d(\rho) \geq 3, D(\rho) = \infty)\\ &\geq \mathbb{P}(E_0 \cap \{D(\rho) = \infty, \Gamma_1 > m\})\\ &\geq \mathbb{P}(E_1) + \mathbb{P}(E_2).\end{aligned}$$

by (4.18), taking $c_5 = c_{10}$, we have

$$\begin{aligned}S(\Gamma_1 > m) &\geq c_5\mathbb{P}\Big(\tau_{\overleftarrow{\rho}} \wedge \tau_{h(\rho)} > m-1 | A_\rho \in [a, \frac{1}{a}]\Big)\\ &= c_5 \sum_{n=1}^{\infty} q_1^n \tilde{\mathbb{P}}_0(\tilde{\tau}_{-1} \wedge \tilde{\tau}_n > m-1 | A_0 \in [a, \frac{1}{a}]).\end{aligned}$$

□

4.4.4 Positive speed on big tree and asymptotic of $|Z_t|$ on small tree

This subsection is devoted to the proof of the following propositions, firstly when the tree is big (i.e. q_1 small), the RWRE has positive speed; when the tree is small (q_1 large), we can compute exactly the asymptotic behavior of $|Z_t|$.

Proposition 4.4.2. *If $q_1\mathbf{E}(A^{-1/2}) < 1$, then*

$$v(\eta) > 0 \text{ and } v(Z) > 0. \tag{4.19}$$

Moreover, we also have $v(Y) > 0$.

Proposition 4.4.3. *Assume that $q_1\mathbf{E}(A^{-1/2}) > 1$, we have $\mathbb{P}$-a.s.*

$$\lim_{n\to\infty}\frac{\log|\eta_n|}{\log n} = \lim_{t\to\infty}\frac{\log|Z_t|}{\log t} = t^* - 1/2 \in (1/2, 1) \tag{4.20}$$

where $t^* = \sup\{t \in \mathbb{R},\ \mathbf{E}(A^t)q_1 \leq 1\}$.

Let us give some definitions and heuristics before proving these propositions, write, for $n \geq 0$,

$$\tau_n(\eta) = \inf\{k \geq 0;\ |\eta_k| = n\} \text{ and } \tau_n(Z) = \inf\{t \geq 0;\ |Z_t| = n\}$$

the hitting times of the n-th generation for η and Z respectively. As a consequence of the law of large numbers, $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty}\frac{\tau_n(\eta)}{n} = \frac{1}{v(\eta)} \text{ and } \lim_{n\to\infty}\frac{\tau_n(Z)}{n} = \frac{1}{v(Z)}.$$

The study of the speed is reduced to the study of $\tau_n(\eta)$ and $\tau_n(Z)$. For any $x \in T$, $n \geq -1$, let N_x and N_n denote the time spent by the walk η at x and at the n-th generation respectively:

$$N(x) = \sum_{k\geq 0} \mathbb{1}_{\eta_k = x}, \quad N_n = \sum_{|x|=n} N(x),$$

observe that

$$\tau_n(\eta) \leq \sum_{k=-1}^{n} N_k, \quad E^{\omega,T}[\tau_n(Z)|\eta] \leq \sum_{x:-1\leq|x|\leq n} N_x \frac{A_x}{1+A_x B_x},$$

where $B_x := \sum_{y:\overleftarrow{y}=x} A_y$.

In what follows, we actually study N_n for large n to show that $\liminf_n \frac{\sum_{k=-1}^{n} N_k}{n} < \infty$, $\mathbb{P}$-a.s. The heuristics is the following. Fix some n_0, K_0 (to choose later), pick some vertex y at the n-th generation, if y roughly lies in a subtree of height n_0 with more than K_0 leaves, then the random walk will immediately go down, thus $\mathbb{E}(N_y)$ will be small c.f. Figure 4.2 left. Otherwise, we seek a down going path $\hat{y}, \dots, y, \dots, \check{y}$ such that every vertex in this path does not branch much except for the two ends, and we need these two ends have more than K_0 descendants after n_0 generations. In such configuration, we can compare the random walk to the one dimensional one, and once the walker reaches one of the ends, it immediately leaves our path $\hat{y}, \dots, \check{y}$ c.f. Figure 4.2 right.

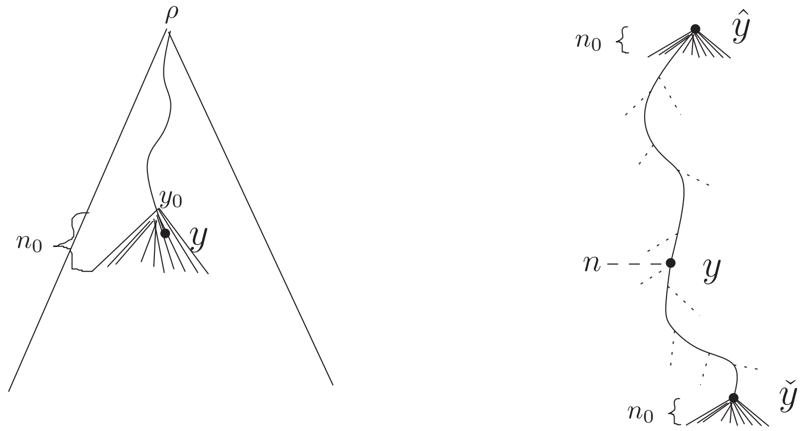

Figure 4.2: Two cases to bound $\mathbb{E}(N_y)$.

If the root have more than K_0 descendants after n_0 generations, then we can always find $\hat{y}$. Otherwise, we need to take n large and use the Galton Watson structure. To handle this issue, let us introduce the following notations. For the GW tree T, let Z_n^T be the number of vertices at the n-th generation. By Lemma 4.1 of [1], we have for any $K_0 \geq 1$,

$$\mathbb{E}_{\mathrm{GW}}(Z_n^T \mathbb{1}_{Z_n^T \leq K_0}) \leq K_0 n^{K_0} q_1^{n-K_0}.$$

Let $r \in (q_1, 1)$ be some real we choose later, let

$$n_0 = n_0(K_0, r) := \inf\{n \geq 1,\ \mathbb{E}_{\mathrm{GW}}(Z_n^T \mathbb{1}_{Z_n^T \leq K_0}) \leq r^n\},$$

which is thus a finite integer. In fact, K_0 will be chosen according to Corollary 4.4.1. Define

$$Z^T(u, n) = |\{x \in T;\ u < x, |x| = |u| + n\}|.$$

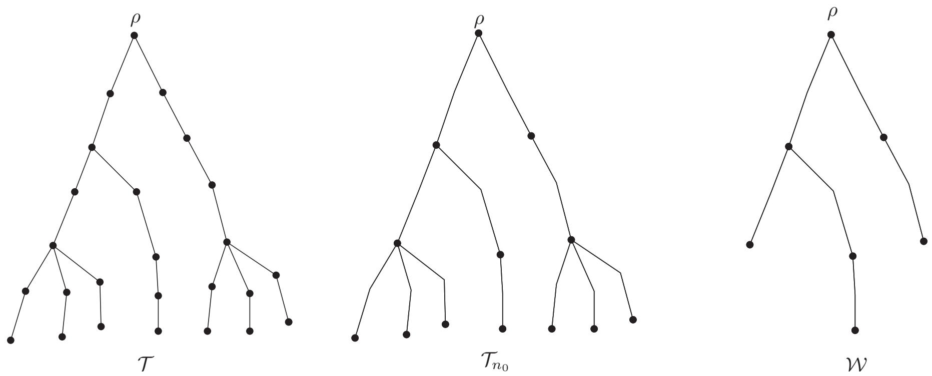

Figure 4.3: An example in the case $K_0 = n_0 = 2$.

Let T_{n_0} be a tree induced from T in the following way: starting from the root ρ, y is a child of x in T_{n_0} if $x < y$ and $|y| = |x| + n_0$. Define a subtree $\mathcal{W}$ of T_{n_0} by

$$\mathcal{W} = \{x \in T_{n_0},\ u < x \Rightarrow Z^T(u, n_0) \le K_0\}.$$

Let W_k be the population of the k-th generation of $\mathcal{W}$, $\mathcal{W}$ is a sub critical Galton Watson tree of mean offspring $\mathbb{E}_{\mathrm{GW}}(Z^T_{n_0} \mathbb{1}_{Z^T_{n_0} \le K_0}) \le r^{n_0}$; in particular, for any $k \ge 0$, $\mathbb{E}_{\mathrm{GW}}(W_k) \le r^{kn_0}$.

For any $y \in T$, let y_0 be the youngest ancestor of y in T_{n_0}. For $n \ge n_0$, let $j = \lfloor \frac{n}{n_0} \rfloor \ge 1$ so that $jn_0 \le n < (j+1)n_0$. Define

$$N_{n,1} = \sum_{|y|=n} N(y) \mathbb{1}_{Z^T(y_0,n_0) > K_0}, \quad N^*_{n,1} = \sum_{|y|=n} N(y) \frac{A_y}{1 + A_y B_y} \mathbb{1}_{Z^T(y_0,n_0) > K_0}. \tag{4.21}$$

$$N_{n,2} = \sum_{|y|=n} N(y) \mathbb{1}_{Z^T(y_0,n_0) \le K_0,\ y_0 \notin \mathcal{W}}, \quad N^*_{n,2} = \sum_{|y|=n} N(y) \frac{A_y}{1 + A_y B_y} \mathbb{1}_{Z^T(y_0,n_0) \le K_0,\ y_0 \notin \mathcal{W}} \tag{4.22}$$

Lemma 4.4.8. *There exist $r \in (q_1, 1)$ and $K_0 > 0$, such that, with the definitions of n_0, $N_{n,1}$, $N^*_{n,1}$ above, for some constant $L > 0$, for any $n \ge n_0$*

$$\mathbb{E}(N_{n,1}) \le L, \quad \mathbb{E}(N^*_{n,1}) \le L. \tag{4.23}$$

Lemma 4.4.9. *With the same assumption as in Lemma 4.4.8, if $0 < \lambda < 1 \wedge (t^* - 1/2)$ where t^* is define in* (4.10)*, then*

$$\mathbb{E}(N^\lambda_{n,2}) \le L, \quad \mathbb{E}((N^*_{n,2})^\lambda) \le L. \tag{4.24}$$

We are prepared to prove Proposition 4.4.2 and Proposition 4.4.3.

Proof of Proposition 4.4.2. Since $q_1 \mathbf{E}(A^{-1/2}) < 1$, $t^* > 3/2$. We choose $\lambda = 1$. As $\mathcal{W}$ is finite a.s., if $\chi = (\text{height}(\mathcal{W}) + 1)n_0$ (where for a finite tree T, $\text{height}(T) := \max_{x \in T} |x|$), then

$$\text{for all } n \ge \chi,\ N_n \le N_{n,1} + N_{n,2}.$$

By Lemma 4.4.8, 4.4.9, for any $n \ge n_0$,

$$\mathbb{E}(N_n, n \ge \chi) \le 2L.$$

Thus,

$$\liminf_{n\to\infty} \mathbb{E}\Big[\frac{\sum_{i=\chi}^{n} N_n}{n}\Big] \le 2L.$$

By Fatou's lemma, a.s.

$$\liminf_{n\to\infty} \frac{\sum_{k=-1}^{n} N_k}{n} = \liminf_{n\to\infty} \frac{\sum_{k=\chi}^{n} N_k}{n} < \infty.$$

Therefore,

$$\frac{1}{v(\eta)} = \liminf_{n\to\infty} \frac{\tau_n}{n} \le \liminf_{n\to\infty} \frac{\sum_{k=-1}^{n} N_k}{n} < \infty.$$

This implies that $v(\eta) > 0$.

The case for Z_t can be treated in a similar manner with N_n^* instead of N_n. Finally, to prove $v(Y) > 0$, it is enough to recall $Z_{D(t)} = Y_t$ where $D(t) = \sum_x (l_x(t)^2 + 2cl_x(t))$ and note that

$$\frac{D(t)}{t} = \frac{\sum_x (l_x(t)^2 + 2cl_x(t))}{\sum_x l_x(t)} \ge 2c > 0.$$

It follows that

$$v(Y) = \lim_{t\to\infty} \frac{|Y_t|}{t} = \lim_{t\to\infty} \frac{|Z_{D(t)}|}{t} \ge v(Z) \liminf_{t\to\infty} \frac{D(t)}{t} \ge 2cv(Z).$$

$\square$

Proof of Proposition 4.4.3. If $q_1 \mathbf{E}(A^{-1/2}) \ge 1$, $\lambda < t^* - 1/2 \le 1$. Let $N_i(Z)$ be the time spent at the i-th generation by (Z_t). Let $\Gamma_k(Z)$ be the regenerative times corresponding to $(Z_t)_{t\ge 0}$. Let $u(n)$ be the unique integer such that $\Gamma_{u(n)} \le \tau_n(Z) < \Gamma_{u(n)+1}$. Then,

$$\begin{aligned}\frac{\Gamma_{u(n)}(Z)^\lambda}{n} \le \frac{\sum_{k\le u(n)} (\Gamma_k(Z) - \Gamma_{k-1}(Z))^\lambda}{n} &= \frac{\sum_{k\le u(n)} (\sum_{i=|Z_{\Gamma_{k-1}(Z)}|}^{i=|Z_{\Gamma_k(Z)}|-1} N_i(Z))^\lambda}{n} \\ &\le \frac{\sum_{i\le n} N_i(Z)^\lambda}{n}.\end{aligned}$$

Taking limit yields that

$$\liminf_{n\to\infty} \frac{\Gamma_{u(n)}(Z)^\lambda}{n} \le \liminf_{n\to\infty} \frac{\sum_{k\le u(n)} (\Gamma_k(Z) - \Gamma_{k-1}(Z))^\lambda}{n} \le \liminf_{n\to\infty} \frac{\sum_{i=\chi}^{n} N_i(Z)^\lambda}{n}.$$

Applying Jensen's inequality then Lemma 4.4.9 implies that

$$\mathbb{E}[N_n(Z)^\lambda; n \ge \chi] \le \mathbb{E}[\mathbf{E}[N_n(Z); n \ge \chi|\eta]^\lambda] \le \mathbb{E}[(N_n^*)^\lambda, n \ge \chi] \le 2L.$$

It follows from Fatou's lemma that

$$\liminf_{n\to\infty} \frac{\Gamma_{u(n)}(Z)^\lambda}{n} \le \liminf_{n\to\infty} \frac{\sum_{k\le u(n)} (\Gamma_k(Z) - \Gamma_{k-1}(Z))^\lambda}{n} < \infty.$$

By law of large numbers,

$$\lim_{n\to\infty} \frac{n}{u(n)} = \mathbb{E}_S[|Z_{\Gamma_1(Z)}|] < \infty, \text{ and } \lim_{n\to\infty} \frac{\sum_{k\le n} (\Gamma_k(Z) - \Gamma_{k-1}(Z))^\lambda}{n} = \mathbb{E}_S[\Gamma_1(Z)^\lambda].$$

Therefore there exists a constant $C \in (0, \infty)$ such that

$$\liminf_{n\to\infty} \frac{\Gamma_n(Z)^\lambda}{n} < C.$$

Note that $|Z_t| \geq \#\{k : \Gamma_k(Z) < t\}$. So we get $|Z_t| \geq t^\lambda / C$ for all sufficiently large t. We hence deduce that

$$\liminf_{t\to\infty} \frac{\log |Z_t|}{\log t} \geq \lambda.$$

Letting $\lambda \uparrow t^* - 1/2$ yields

$$\liminf_{n\to\infty} \frac{\log |Z_t|}{\log t} \geq t^* - 1/2. \tag{4.25}$$

The result follows by Remark 4.4.1. Similar arguments can be applied to $\lim_{n\to\infty} \frac{\log |\eta_n|}{\log n}$. □

It remains to show the main Lemmas 4.4.8,4.4.9. Let us first state some preliminary results. As the walk is transient, the support of the random walk should be slim. This is formulated in the following lemma:

Lemma 4.4.10. *There exists a constant $c_{11} > 0$ such that for any $n \geq 1$, $\mathbb{E}(\sum_{|x|=n} \mathbb{1}_{\tau_x<\infty}) \leq c_{11}$.*

The following lemma shows that, the escape probability is relatively large.

Lemma 4.4.11. *Consider i.i.d. copies of GW trees $T^{(i)}$ rooted at $\rho^{(i)}$ with independent environment $\omega^{(i)}$, for each $T^{(i)}$, define $\beta_i = P^{\omega^{(i)},T^{(i)}}_{\rho^{(i)}}(\tau_{\overleftarrow{\rho^{(i)}}} = \infty)$. There exists an integer $K = K(q_1, c) \geq 1$ such that*

$$\mathbb{E}(\frac{1}{\sum_{i=1}^K \beta_i}) \leq c_{12} < \infty \text{ and } \mathbb{E}(\frac{1}{\sum_{i=1}^K A_{\rho^{(i)}}\beta_i}) < c_{12} < \infty.$$

Moreover, if $q_1\xi_2 < 1$, then $\mathbb{E}(\frac{1}{\beta(\rho)}) \leq c_{12} < \infty$ and $\mathbb{E}(\frac{1}{A_\rho \beta(\rho)}) < c_{12} < \infty$.

Remark 4.4.2. *In fact, if $q_1\mathbf{E}(A^{-2}) < 1$, a proof similar to Proposition 2.3 of [1] shows that η has positive speed, in particular, the VRJP on any regular tree (except $\mathbb{Z}$) admits positive speed.*

Corollary 4.4.1. *There exists $K_0 \geq K$, such that*

$$\mathbb{E}(\frac{1}{\sum_1^{K_0} A^2_{\rho^{(i)}}\beta_i^2}) < c_{13} < \infty.$$

The proof of Lemma 4.4.10, 4.4.11 and Corollary 4.4.1 will be postponed to the Section 4.6, let us state the consequence of these preliminary results. Recall that Z_n^T is the population at generation n, and that for any $x \in T$, τ_x is the first hitting time, τ_x^* the first return time to x. For $u, v \in T$ write $u < v$ if u is an ancestor of v and define

$$p_1(u, v) = P_u^{\omega,T}(\tau_{\overleftarrow{u}} = \infty, \tau_u^* = \infty, \tau_v = \infty)$$

Lemma 4.4.12. *For any $n \geq 2$ and $k \in \{1, 2\}$, consider K_0 as in Corollary 4.4.1, we have*

$$\mathbb{E}\Big(\mathbb{1}_{Z_n^T > K_0} \sum_{|u|=n} \frac{1}{p_1(\rho, u)^k}\Big) < c_{14}^n < \infty.$$

In addition,

$$\mathbb{E}\Big(\mathbb{1}_{Z_n^T > K_0} \sum_{|u|=n} \frac{1}{p_1(\rho, u)^k}\Big|A_\rho\Big) < c_{14}^n(1 + \frac{1}{A_\rho}). \tag{4.26}$$

Proof of Lemma 4.4.12. Fix $n \geq 2$, let $\Upsilon_0 := \inf\{l \geq 1;\ Z_l > K_0\}$, then $\{Z_n^T > K_0\} = \{\Upsilon_0 \leq n\}$. For any $u \in T$ such that $|u| \geq \Upsilon_0$, let U be its ancestor at the Υ_0-th generation. By Markov property,

$$\begin{aligned} p_1(\rho, u) &\geq \sum_{|y|=\Upsilon_0-1} P_\rho^{\omega,T}(\tau_y < \tau_\rho^*) P_y^{\omega,T}(\tau_{\overleftarrow{y}} = \infty, \tau_U = \infty) \\ &\geq \sum_{|y|=\Upsilon_0-1} \prod_{i=0}^{\Upsilon_0-2} p(y_i, y_{i+1}) P_y^{\omega,T}(\tau_{\overleftarrow{y}} = \infty, \tau_U = \infty) \end{aligned} \tag{4.27}$$

where $\{y_0(=\rho), y_1, \dots, y_{\Upsilon_0-1}(=y)\}$ is the unique path connecting ρ and y. Note that if $\overleftarrow{U} = y$, then

$$P_y^{\omega,T}(\tau_{\overleftarrow{y}} = \infty, \tau_U = \infty) = \sum_{z:\, \overleftarrow{z}=y, z\neq U} p(y,z)\beta(z) + \sum_{z:\, \overleftarrow{z}=y, z\neq U} p(y,z)(1-\beta(z)) P_y^{\omega,T}(\tau_{\overleftarrow{y}} = \infty, \tau_U = \infty).$$

Otherwise

$$P_y^{\omega,T}(\tau_{\overleftarrow{y}} = \infty, \tau_U = \infty) = \sum_{z:\, \overleftarrow{z}=y} p(y,z)\beta(z) + \sum_{z:\, \overleftarrow{z}=y} p(y,z)(1-\beta(z)) P_y^{\omega,T}(\tau_{\overleftarrow{y}} = \infty, \tau_U = \infty)$$

It follows that in both cases,

$$\begin{aligned} P_y^{\omega,T}(\tau_{\overleftarrow{y}} = \infty, \tau_U = \infty) &= \frac{\sum_{z:\, \overleftarrow{z}=y} \mathbb{1}_{z\neq U}\, p(y,z)\beta(z)}{p(y,\overleftarrow{y}) + p(y,U) + \sum_{z:\, \overleftarrow{z}=y} \mathbb{1}_{z\neq U}\, p(y,z)\beta(z)} \\ &\geq \frac{\sum_{z:\, \overleftarrow{z}=y} \mathbb{1}_{z\neq U} A_y A_z \beta(z)}{1 + A_y A_U + \sum_{z:\, \overleftarrow{z}=y} \mathbb{1}_{z\neq U} A_y A_z \beta(z)} \\ &\geq \frac{A_y}{1+A_y} \frac{1}{1+A_U} \frac{\sum_{z:\, \overleftarrow{z}=y} \mathbb{1}_{z\neq U} A_z \beta(z)}{1 + \sum_{z:\, \overleftarrow{z}=y} \mathbb{1}_{z\neq U} A_z \beta(z)} \end{aligned}$$

Plugging it into (4.27) yields that

$$\begin{aligned} p_1(\rho, u) &\geq \sum_{|y|=\Upsilon_0-1} \prod_{i=0}^{\Upsilon_0-2} p(y_i, y_{i+1}) \frac{A_y}{1+A_y} \frac{1}{1+A_U} \frac{\sum_{z:\, \overleftarrow{z}=y} \mathbb{1}_{z\neq U} A_z \beta(z)}{1 + \sum_{z:\, \overleftarrow{z}=y} \mathbb{1}_{z\neq U} A_z \beta(z)} \\ &\geq \frac{1}{1+A_U} \min_{|y|=\Upsilon_0-1} \left(\prod_{i=0}^{\Upsilon_0-2} p(y_i, y_{i+1}) \frac{A_y}{1+A_y} \right) \cdot \frac{\sum_{z:|z|=\Upsilon_0, z\neq U} A_z \beta(z)}{1 + \sum_{z:|z|=\Upsilon_0, z\neq U} A_z \beta(z)} \end{aligned}$$

Thus, for $k \in \{1, 2\}$,

$$\frac{1}{p_1(\rho, u)^k} \leq (1+A_U)^k \frac{1}{\min_{|y|=\Upsilon_0-1} \left(\prod_{i=0}^{\Upsilon_0-2} p(y_i, y_{i+1}) \frac{A_y}{1+A_y} \right)^k} \left(1 + \frac{1}{\sum_{z:|z|=\Upsilon_0, z\neq U} A_z \beta(z)} \right)^k.$$

Given the tree T, by integrating w.r.t. $\mathbf{P}(d\omega)$, we have

$$\begin{aligned} \mathbb{1}_{n\geq\Upsilon_0} \sum_{|u|=n} \mathbb{E}^T\left(\frac{1}{p_1(\rho,u)^k} \right) &\leq \mathbb{E}^T \left(\frac{1}{\min_{|y|=\Upsilon_0-1} \left(\prod_{i=0}^{\Upsilon_0-2} p(y_i, y_{i+1}) \frac{A_y}{1+A_y} \right)^k} \right) \\ &\quad \times \sum_{|U|=\Upsilon_0} Z^T(U, n-\Upsilon_0) \mathbb{E}^T[(1+A_U)^k] \mathbb{E}^T \left(\left(1 + \frac{1}{\sum_{z:|z|=\Upsilon_0, z\neq U} A_z \beta(z)} \right)^k \right) \end{aligned}$$

It follows from Lemma 4.4.11 that

$$\begin{aligned}
&\mathbf{E_Q}\left(\mathbb{1}_{n\geq\Upsilon_0}\sum_{|u|=n}\frac{1}{p_1(\rho,u)^k}\middle|\Upsilon_0, Z_k; 0\leq k\leq \Upsilon_0\right)\\
\leq& c_{15}\mathbb{1}_{n\geq\Upsilon_0}\mathbb{E}^T\left(\frac{1}{\min_{|y|=\Upsilon_0-1}\left(\prod_{i=0}^{\Upsilon_0-2}p(y_i,y_{i+1})\frac{A_y}{1+A_y}\right)^k}\right)\times\sum_{|U|=\Upsilon_0}\mathbf{E}[(1+A)^k]b^{n-\Upsilon_0}\\
\leq& c_{16}\mathbb{1}_{n\geq\Upsilon_0}\sum_{|y|=\Upsilon_0-1}\mathbb{E}^T\left[\left(\prod_{i=0}^{\Upsilon_0-2}\frac{(1+A_{y_i})(1+B_{y_i})}{A_{y_i}A_{y_{i+1}}}\frac{1+A_y}{A_y}\right)^k\right]\sum_{|U|=\Upsilon_0}b^{n-\Upsilon_0}.
\end{aligned}$$

By independence of $A_x, x\in T$, we see that

$$\mathbb{E}^T\left[\left(\prod_{i=0}^{\Upsilon_0-2}\frac{(1+A_{y_i})(1+B_{y_i})}{A_{y_i}A_{y_{i+1}}}\frac{1+A_y}{A_y}\right)^k\right]\leq c_{17}^{\Upsilon_0-1},$$

with $c_{17}\in(1,\infty)$. Consequently,

$$\begin{aligned}
\mathbf{E_Q}\left(\mathbb{1}_{n\geq\Upsilon_0}\sum_{|u|=n}\frac{1}{p_1(\rho,u)^k}\right)\leq& \mathbf{E_Q}\Big(c_{16}\mathbb{1}_{n\geq\Upsilon_0}\sum_{|y|=\Upsilon_0-1}c_{15}^{\Upsilon_0-1}\sum_{|U|=\Upsilon_0}b^{n-\Upsilon_0}\Big)\\
\leq& c_{16}K_0\mathbf{E_Q}\Big(\mathbb{1}_{n\geq\Upsilon_0}c_{15}^{n-1}Z_n^T\Big)\\
\leq& c_{18}(c_{17}b)^n<\infty.
\end{aligned}$$

(4.26) follows in the same way. This completes the proof. □

Proof of Lemma 4.4.8. We only bound $\mathbb{E}(N_{n,1})$, the argument for $\mathbb{E}(N^*_{n,1})$ is similar. For any $y\in T$ at the n-th generation such that $Z^T(y_0,n_0)>K_0$, let Y be the youngest ancestor of y such that $Z^T(Y,n_0)>K_0$. Clearly, $y_0\leq Y\leq y$. So,

$$N_{n,1}=\sum_{|y|=n}N(y)\mathbb{1}_{Z^T(y_0,n_0)>K_0}\leq\sum_{|y|=n}N(y)\mathbb{1}_{y_0\leq Y\leq y}.$$

Taking expectation w.r.t. $E_\rho^{\omega,T}$ implies that

$$E^{\omega,T}(N_{n,1})\leq\sum_{|y|=n}E^{\omega,T}(N(y))\mathbb{1}_{y_0\leq Y\leq y}=\sum_{|y|=n}P^{\omega,T}(\tau_y<\infty)E_y^{\omega,T}(N(y))\mathbb{1}_{y_0\leq Y\leq y}.$$

Applying the Markov property at τ_Y to $E_y^{\omega,T}(N(y))$, we have

$$E_y^{\omega,T}(N(y))=G^{\tau_Y}(y,y)+P_y^{\omega,T}(\tau_Y<\infty)P_Y^{\omega,T}(\tau_y<\infty)E_y^{\omega,T}(N(y))$$

where (write $\{(\tau_Y\wedge\infty)>\tau_y^*\}=\{\tau_y^*<\infty \text{ and } \tau_y^*<\tau_Y\}$ for short)

$$G^{\tau_Y}(y,y)=E_y^{\omega,T}\Big(\sum_{k=0}^{\tau_Y}\mathbb{1}_{\eta_k=y}\Big)=\frac{1}{1-P_y^{\omega,T}((\tau_Y\wedge\infty)>\tau_y^*)}.$$

Hence

$$E_y^{\omega,T}(N(y)) = \frac{G^{\tau_Y}(y,y)}{1 - P_Y^{\omega,T}(\tau_y < \infty) P_y^{\omega,T}(\tau_Y < \infty)} \le \frac{G^{\tau_Y}(y,y)}{1 - P_Y^{\omega,T}(\tau_Y^* < \infty)} = \frac{G^{\tau_Y}(y,y)}{P_Y^{\omega,T}(\tau_Y^* = \infty)}.$$

We bound $G^{\tau_Y}(y,y)$ first. As $P_y^{\omega,T}((\tau_Y \wedge \infty) > \tau_y^*) \le \sum_{z:\overleftarrow{z}=y} p(y,z) + p(y,\overleftarrow{y}) P_{\overleftarrow{y}}^{\omega,T}(\tau_y < (\tau_Y \wedge \infty))$,

$$1 - P_y^{\omega,T}((\tau_Y \wedge \infty) > \tau_y^*) \ge p(y,\overleftarrow{y})\Big(1 - P_{\overleftarrow{y}}^{\omega,T}(\tau_y < \tau_Y)\Big).$$

By Lemma 4.4 of [1] and (4.38), the right hand side of the above inequality is larger than

$$p(y,\overleftarrow{y})\Big(1 - \tilde{P}_{\overleftarrow{y}}^{\omega,T}(\tilde{\tau}_y < \tilde{\tau}_Y)\Big) = \frac{1}{1 + A_y B_y} \frac{1}{1 + A_y \sum_{Y<z<y} A_z \prod_{z<u<y} A_u^2}.$$

where we identify $\tilde{P}_{\overleftarrow{y}}^{\omega,T}$ to the probability of $(\tilde{\eta}_n)$ on the segment $[\![Y, y]\!]$. Therefore,

$$G^{\tau_Y}(y,y) \le \Big(1 + A_y \sum_{Y<z<y} A_z \prod_{z<u<y} A_u^2\Big)(1 + A_y B_y) =: V_{y,Y}.$$

Consequently,

$$E^{\omega,T}(N(y)) \mathbb{1}_{Z^T(y_0,n_0)>K_0} \le P^{\omega,T}(\tau_Y < \infty) \frac{V_{y,Y}}{P_Y^{\omega,T}(\tau_Y^* = \infty)} \mathbb{1}_{Z^T(Y,n_0)>K_0,\ y_0 \le Y \le y}.$$

Summing over all possibilities of Y yields that (recall that $j = \lfloor \frac{n}{n_0} \rfloor$)

$$\begin{aligned} E^{\omega,T}(N_{n,1}) &\le \sum_{l=jn_0}^{n} \sum_{|Y|=l} P^{\omega,T}(\tau_Y < \infty) \frac{\sum_{|y|=n, Y\le y} V_{y,Y}}{P_Y^{\omega,T}(\tau_Y^* = \infty)} \mathbb{1}_{Z^T(Y,n_0)>K_0} \\ &\le \sum_{l=jn_0}^{n} \sum_{|Y|=l} P^{\omega,T}(\tau_{\overleftarrow{Y}} < \infty) \frac{\sum_{|y|=n, Y\le y} V_{y,Y}}{P_Y^{\omega,T}(\tau_Y^* = \infty, \tau_{\overleftarrow{Y}} = \infty)} \mathbb{1}_{Z^T(Y,n_0)>K_0}, \end{aligned}$$

where the last inequality holds because $P^{\omega,T}(\tau_Y < \infty) \le P^{\omega,T}(\tau_{\overleftarrow{Y}} < \infty)$ and $P_Y^{\omega,T}(\tau_Y^* = \infty) \ge P_Y^{\omega,T}(\tau_Y^* = \infty, \tau_{\overleftarrow{Y}} = \infty)$. Summing over the value of $\overleftarrow{Y}$ yields that

$$E^{\omega,T}(N_{n,1}) \le \sum_{l=jn_0-1}^{n-1} \sum_{|x|=l} P^{\omega,T}(\tau_x < \infty) \sum_{Y:\overleftarrow{Y}=x} \frac{\sum_{|y|=n, Y\le y} V_{y,Y}}{P_Y(\tau_Y^* = \infty, \tau_{\overleftarrow{Y}} = \infty)} \mathbb{1}_{Z^T(Y,n_0)>K_0}.$$

As conditionally on T, $P^{\omega,T}(\tau_x < \infty)$ and $\sum_{Y:\overleftarrow{Y}=x} \frac{\sum_{|y|=n, Y\le y} V_{y,Y}}{P_Y(\tau_Y^*=\infty, \tau_{\overleftarrow{Y}}=\infty)} \mathbb{1}_{d(Y,n_0)>K_0}$ are independent,

$$\begin{aligned} \mathbb{E}(N_{n,1}) &\le \mathbb{E}\left(\sum_{l=jn_0-1}^{n-1} \sum_{|x|=l} \mathbb{E}^T(P^{\omega,T}(\tau_x < \infty)) \mathbb{E}^T\Big(\sum_{Y:\overleftarrow{Y}=x} \frac{\sum_{|y|=n, Y\le y} V_{y,Y}}{P_Y^{\omega,T}(\tau_Y^* = \infty, \tau_{\overleftarrow{Y}} = \infty)} \mathbb{1}_{Z^T(Y,n_0)>K_0}\Big)\right) \\ &= \sum_{l=jn_0-1}^{n-1} \mathbb{E}(\sum_{|x|=l} \mathbb{1}_{\tau_x<\infty}) \mathbb{E}\Big(\sum_{|Y|=1} \frac{\sum_{|y|=n-l, Y\le y} V_{y,Y}}{P_Y^{\omega,T}(\tau_Y^* = \infty, \tau_{\overleftarrow{Y}} = \infty)} \mathbb{1}_{Z^T(Y,n_0)>K_0}\Big) \end{aligned}$$

Note that for any $|Y|=1$, $\frac{\sum_{|y|=n-l, Y\le y} V_{y,Y}}{P_Y^{\omega,T}(\tau_Y^*=\infty, \tau_{\overleftarrow{Y}}=\infty)} \mathbb{1}_{Z^T(Y,n_0)>K_0}$ are i.i.d. By Lemma 4.4.10,

$$\mathbb{E}(N_{n,1}) \le bc_{11} \sum_{l=jn_0-1}^{n-1} \mathcal{A}_{n-l} \tag{4.28}$$

where

$$\mathcal{A}_{n-l} = \mathbb{E}\Big(\frac{\sum_{|y|=n-l-1} V_{y,\rho}}{P^{\omega,T}(\tau_\rho^* = \infty, \tau_{\overleftarrow{\rho}} = \infty)} \mathbb{1}_{Z^T(\rho,n_0)>K_0} \Big).$$

By Cauchy-Schwartz inequality,

$$\mathcal{A}_{n-l} \le \mathbb{E}\Big[\Big(\sum_{|y|=n-l-1} V_{y,\rho} \Big)^2\Big] \mathbb{E}\Big[\frac{\mathbb{1}_{Z^T(\rho,n_0)>K_0}}{P^{\omega,T}(\tau_\rho^* = \infty, \tau_{\overleftarrow{\rho}} = \infty)^2} \Big]$$

Recall that Z_n^T denote the number of vertices at the n-th generation of the tree T, using Lemma 4.4.12 then Applying again Cauchy-Schwartz inequality to $\Big(\sum_{|y|=n-l-1} V_{y,\rho} \Big)^2$ implies that

$$\begin{aligned}
\mathcal{A}_{n-l} &\le c_{14}^{n_0} \mathbb{E}\Big(Z_{n-l-1}^T \sum_{|y|=n-l-1} V_{y,\rho}^2 \Big) \\
&\le c_{19} \mathbf{E}_{GW}[c_{20}^{n-l-1} \Big(Z_{n-l-1}^T \Big)^2],
\end{aligned}$$

where the second inequality follows from $\mathbb{E}^T[V_{y,\rho}] \le c_{20}^{|y|}$. Plugging it into (4.28) implies that

$$\mathbb{E}(N_{n,1}) \le bc_{11}c_{19} \sum_{l=jn_0-1}^{n-1} \mathbf{E}_{GW}[c_{20}^{n-l-1} \Big(Z_{n-l-1}^T \Big)^2] \le c_{21} \sum_{k=0}^{n_0} c_{20}^k \mathbf{E}_{GW}\Big[(Z_k^T)^2 \Big] \le c_{22},$$

since $\mathbf{E}_{GW}[(Z_1^T)^2] < \infty$. Analoguesly, for $N_{n,1}^*$ we get that

$$E^{\omega,T}(N_{n,1}^*) \le \sum_{l=jn_0-1}^{n-1} \sum_{|x|=l} P^{\omega,T}(\tau_x < \infty) \sum_{Y: \overleftarrow{Y}=x} \frac{\sum_{|y|=n, Y\le y} V_{y,Y} \frac{A_y}{1+A_yB_y}}{P_Y^{\omega,T}(\tau_Y^* = \infty, \tau_{\overleftarrow{Y}} = \infty)} \mathbb{1}_{Z^T(Y,n_0)>K_0}.$$

And recounting on the same arguments gives a finite upper bound for $\mathbb{E}[N_{n,1}^*]$. □

Proof of Lemma 4.4.9. Again we only give the proof for $\mathbb{E}(N_{n,2}^\lambda)$. For $y \in T$, as $Z^T(y_0, n_0) \le K_0$ and $y_0 \notin \mathcal{W}$, we can find the youngest ancestor Y_1 of y in T_{n_0} such that $Z^T(Y_1, n_0) > K_0$, automatically $Y_1 < y_0$. Let Y_2 be the youngest descendant of Y_1 in T_{n_0} such that it is an ancestor of y. Let Y_3 be the youngest descendant of y in T_{n_0} such that $Z^T(Y_3, n_0) > K_0$.
For any $0 < \lambda \le 1$,

$$\begin{aligned}
E^{\omega.T}[N_{n,2}^\lambda] &\le E^{\omega,T}\Big[\sum_{|y|=n} N(y)^\lambda \mathbb{1}_{Z^T(y_0,n_0)\le K_0,\ y_0\notin\mathcal{W}} \Big] \\
&\le \sum_{|y|=n} \mathbb{1}_{Z^T(y_0,n_0)\le K_0,\ y_0\notin\mathcal{W}} P^{\omega,T}(\tau_y < \infty) \Big(E_y^{\omega,T}[N(y)] \Big)^\lambda .
\end{aligned} \tag{4.29}$$

In what follows, we identify $\widetilde{P}^\omega$ with the distribution of a one-dimensional random walk $\widetilde{\eta}$ on the path $[\![\overleftarrow{Y_1},\ Y_3]\!]$. Let us state the following lemmas which will be used in (4.29).

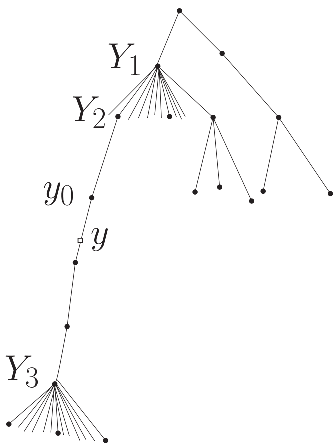

Figure 4.4: An example of Y_1, Y_2, Y_3.

Lemma 4.4.13. *For any $y \in T$ such that $Y_1 < Y_2 < y < Y_3$, let y^* be the unique child of y which is also ancestor of Y_3. Then,*

$$\left(E_y^{\omega,T}[N(y)]\right)^\lambda \le \left(\frac{1+A_yB_y}{1+A_yA_{y^*}}\right)^\lambda \widetilde{G}^{\tilde\tau_{Y_1}\wedge\tilde\tau_{Y_3}}(y,y)^\lambda \frac{2}{p_1(Y_1,Y_2)P^{\omega,T}_{Y_3}(\tau^*_{Y_3}=\infty,\tau_{\overleftarrow{Y_3}}=\infty)}. \tag{4.30}$$

where $\widetilde{G}^{\tilde\tau_{Y_1}\wedge\tilde\tau_{Y_3}}(y,y)=\widetilde{E}^\omega_y\left(\sum_{k=0}^{\tilde\tau_{Y_1}\wedge\tilde\tau_{Y_3}}\mathbb{1}_{\widetilde{\eta}_k=y}\right)$ is the Green function associated with $(\widetilde{\eta}_n)$.

Lemma 4.4.14.

$$P^{\omega,T}(\tau_y<\infty)\le P^{\omega,T}(\tau_{Y_1}<\infty)\widetilde{P}^\omega_{Y_1}(\tilde\tau_y<\tilde\tau_{Y_1-1})^\lambda\frac{1}{p_1(Y_1,Y_2)}. \tag{4.31}$$

The proofs of Lemmas 4.4.13 and 4.4.14 can be found in section 5.2 of [1] with slight modifications, so we feel free to omit them (see (5.10) and (5.11) therein). Now plugging (4.30) and (4.31) into (4.29) yields that

$$E^{\omega,T}(N^\lambda_{n,2})\le\sum_{|y|=n}\frac{2P^{\omega,T}(\tau_{Y_1}<\infty)}{p_1(Y_1,Y_2)^2P^{\omega,T}_{Y_3}(\tau^*_{Y_3}=\infty,\tau_{\overleftarrow{Y_3}}=\infty)}\left(\frac{1+A_yB_y}{1+A_yA_{y^*}}\widetilde{P}^\omega_{Y_1}(\tilde\tau_y<\tilde\tau_{Y_1-1})\widetilde{G}^{\tilde\tau_{Y_1}\wedge\tilde\tau_{Y_3}}(y,y)\right)^\lambda.$$

By Lemma 4.4.4, one sees that

$$\begin{aligned}E^{\omega,T}(N^\lambda_{n,2})&\le\sum_{|y|=n}\frac{2P^{\omega,T}(\tau_{Y_1}<\infty)}{p_1(Y_1,Y_2)^2P^{\omega,T}_{Y_3}(\tau^*_{Y_3}=\infty,\tau_{\overleftarrow{Y_3}}=\infty)}\left(\frac{1+A_yB_y}{1+A_yA_{y^*}}\widetilde{E}^\omega_{Y_1}[\tilde\tau_{\overleftarrow{Y_1}}\wedge\tilde\tau_{Y_3}]\right)^\lambda\\&\le\sum_{|y|=n}\frac{2P^{\omega,T}(\tau_{Y_1}<\infty)}{p_1(Y_1,Y_2)^2P^{\omega,T}_{Y_3}(\tau^*_{Y_3}=\infty,\tau_{\overleftarrow{Y_3}}=\infty)}\left(\frac{1+A_yB_y}{1+A_yA_{y^*}}\right)^\lambda S_{\lambda,[\![Y_1,Y_2]\!]}\left(1+A^\lambda_{Y_2^*}\left(1+\widetilde{E}^\omega_{Y_2^*}[\tilde\tau_{Y_2}\wedge\tilde\tau_{Y_3}]^\lambda\right)\right)\end{aligned}$$

where Y_2^* is the children of Y_2 along $[\![Y_2,Y_3]\!]$. Decompose the sum over $|y|=n$ by

$$\sum_{|y|=n}=\sum_{y:|y|=n,Y_1=\rho}+\sum_{l=1}^{(j-1)}\sum_{|x|=ln_0-1}\sum_{y:\overleftarrow{Y_1}=x,|y|=n}.$$

We get that

$$E^{\omega,T}(N^{\lambda}_{n,2}) \le \sum_{|y|=n, Y_1=\rho} \frac{2S_{\lambda,[\![\rho,Y_2]\!]}}{p_1(\rho,Y_2)^2 P^{\omega,T}_{Y_3}(\tau^*_{Y_3}=\infty, \tau_{\overleftarrow{Y_3}}=\infty)} \Theta_\lambda(Y_2,y,Y_3)$$
$$+\sum_{l=1}^{j-1}\sum_{|x|=ln_0-1}\sum_{|y|=n, \overleftarrow{Y_1}=x} \frac{2P^{\omega,T}(\tau_{Y_1}<\infty) S_{\lambda,[\![Y_1,Y_2]\!]}}{p_1(Y_1,Y_2)^2 P^{\omega,T}_{Y_3}(\tau^*_{Y_3}=\infty, \tau_{\overleftarrow{Y_3}}=\infty)} \Theta_\lambda(Y_2,y,Y_3),$$

where

$$\Theta_\lambda(Y_2,y,Y_3) := \left(\frac{1+A_yB_y}{1+A_yA_{y^*}}\right)^\lambda \left(1 + A^\lambda_{Y_2^*}\left(1+\widetilde{E}^\omega_{Y_2^*}[\tilde\tau_{Y_2}\wedge\tilde\tau_{Y_3}]^\lambda\right)\right).$$

Given the GW tree T, note that $S_{\lambda,[\![Y_1,Y_2]\!]} \in \sigma\{A_z; Y_1 \le z \le Y_2\}$, $p_1(\rho,Y_2) \in \sigma\{A_u : u \in (T\setminus T_{Y_2}) \cup \{Y_2\}\}$, $P^{\omega,T}_{Y_3}(\tau^*_{Y_3}=\infty, \tau_{\overleftarrow{Y_3}}=\infty) \in \sigma\{A_u; u\in T_{Y_3}\}$ and $\Theta_\lambda(Y_2,y,Y_3)\in\sigma\{A_u; Y_2<u\le Y_3\}$. Therefore,

$$\mathbb{E}^T[N^\lambda_{n,2}] \le \sum_{|y|=n,Y_1=\rho} \mathbb{E}^T\left[\frac{2S_{\lambda,[\![\rho,Y_2]\!]}}{p_1(\rho,Y_2)^2}\right] \mathbb{E}^T\left[\frac{\Theta_\lambda(Y_2,y,Y_3)\mathbb{1}_{Z^T(Y_3,n_0)>K_0}}{P^{\omega,T}_{Y_3}(\tau^*_{Y_3}=\infty,\tau_{\overleftarrow{Y_3}}=\infty)}\right]$$
$$+\sum_{l=1}^{j-1}\sum_{|x|=ln_0-1}\sum_{|y|=n,\overleftarrow{Y_1}=x} \mathbb{E}^T\left[\frac{2P^{\omega,T}(\tau_{Y_1}<\infty)S_{\lambda,[\![Y_1,Y_2]\!]}}{p_1(Y_1,Y_2)^2}\right]\mathbb{E}^T\left[\frac{\Theta_\lambda(Y_2,y,Y_3)\mathbb{1}_{Z^T(Y_3,n_0)>K_0}}{P^{\omega,T}_{Y_3}(\tau^*_{Y_3}=\infty,\tau_{\overleftarrow{Y_3}}=\infty)}\right]. \tag{4.32}$$

Observe that

$$P^{\omega,T}_{Y_3}(\tau^*_{Y_3}=\infty,\tau_{\overleftarrow{Y_3}}=\infty) \ge p_1(Y_3,u)\mathbb{1}_{Y_3<u,|u|=|Y_3|+n_0}.$$

$$\mathbb{E}^T\left[\frac{\Theta_\lambda(Y_2,y,Y_3)\mathbb{1}_{Z^T(Y_3,n_0)>K_0}}{P^{\omega,T}_{Y_3}(\tau^*_{Y_3}=\infty,\tau_{\overleftarrow{Y_3}}=\infty)}\middle| A_u, Y_2<u\le Y_3\right] = \Theta_\lambda(Y_2,y,Y_3)\mathbb{E}\left[\frac{\mathbb{1}_{Z^T(Y_3,n_0)>K_0}}{P^{\omega,T}_{Y_3}(\tau^*_{Y_3}=\infty,\tau_{\overleftarrow{Y_3}}=\infty)}\middle| A_{Y_3}\right]$$
$$\le \Theta_\lambda(Y_2,y,Y_3)\mathbb{E}\left[\mathbb{1}_{Z^T(Y_3,n_0)>K_0}\sum_{u:Y_3<u,|u|=|Y_3|+n_0}\frac{1}{p_1(Y_3,u)}\middle| A_{Y_3}\right].$$

Applying Lemma 4.4.12 to the subtree rooted at Y_3 implies that

$$\mathbb{E}^T\left[\frac{\Theta_\lambda(Y_2,y,Y_3)\mathbb{1}_{Z^T(Y_3,n_0)>K_0}}{P^{\omega,T}_{Y_3}(\tau^*_{Y_3}=\infty,\tau_{\overleftarrow{Y_3}}=\infty)}\right] \le c_{23}\mathbb{E}^T\left[(1+\frac{1}{A_{Y_3}})\Theta_\lambda(Y_2,y,Y_3)\right].$$

Plugging it into (4.32) implies that

$$\mathbb{E}^T[N^\lambda_{n,2}] \le \Delta_1(n)+\Delta_2(n),$$

where

$$\Delta_1(n) := 2c_{23}\sum_{|y|=n,Y_1=\rho}\mathbb{E}^T\left[\frac{S_{\lambda,[\![\rho,Y_2]\!]}}{p_1(\rho,Y_2)^2}\right]\mathbb{E}^T\left[(1+\frac{1}{A_{Y_3}})\Theta_\lambda(Y_2,y,Y_3)\right] \tag{4.33}$$
$$\Delta_2(n) := 2c_{23}\sum_{l=1}^{j-1}\sum_{|x|=ln_0-1}\sum_{|y|=n,\overleftarrow{Y_1}=x}\mathbb{E}^T\left[\frac{P^{\omega,T}(\tau_{Y_1}<\infty)S_{\lambda,[\![Y_1,Y_2]\!]}}{p_1(Y_1,Y_2)^2}\right]\mathbb{E}^T\left[(1+\frac{1}{A_{Y_3}})\Theta_\lambda(Y_2,y,Y_3)\right]. \tag{4.34}$$

So,

$$\mathbb{E}[N^{\lambda}_{n,2}] \le \mathbf{E}_{\mathbf{Q}}[\Delta_1(n) + \Delta_2(n)]. \tag{4.35}$$

We firstly bound $\Delta_1(n)$, note that (since $\lambda \le 1$)

$$\Big(\frac{1 + A_y B_y}{1 + A_y A_{y^*}}\Big)^{\lambda} \le \Big(1 + \frac{\sum_{z:\overleftarrow{z}=y, z\neq y^*} A_z}{A_{y^*}}\Big)^{\lambda} \le 1 + \frac{\sum_{z:\overleftarrow{z}=y, z\neq y^*} A_z^{\lambda}}{A_{y^*}^{\lambda}},$$

with $\sum_{z:\overleftarrow{z}=y, z\neq y^*} 1 \le K_0$. If $|Y_2| = mn_0 < n$, $|Y_3| = (m+k)n_0 > n$, by Markov property and the fact that $\{A_z, \overleftarrow{z} = y, z \neq y^*\}$ is independent of $\{A_z, z \in [\![Y_2, Y_3]\!] := [\![-1, kn_0 - 1]\!]\}$

$$\begin{aligned}
&\mathbb{E}^T\Big[(1 + \frac{1}{A_{Y_3}})\Theta_\lambda(Y_2, y, Y_3)\Big] \\
&\le \mathbb{E}^T\Bigg[(1 + \frac{1}{A_{kn_0-1}})(1 + \frac{\sum_{z:\overleftarrow{z}=y, z\neq y^*} A_z^{\lambda}}{A^{\lambda}_{n-mn_0}})(1 + A_0^{\lambda}(1 + \tilde{E}_0^{\omega}(\tilde{\tau}_{-1} \wedge \tilde{\tau}_{kn_0-1})^{\lambda}))\Bigg] \\
&\le c_{24} + c_{24}\mathbf{E}\Big((1 + \frac{1}{A_{kn_0-1}})(1 + \frac{1}{A^{\lambda}_{n-mn_0}})A_0^{\lambda}\tilde{E}_0^{\omega}[\tilde{\tau}_{-1} \wedge \tilde{\tau}_{kn_0-1}]^{\lambda}\Big).
\end{aligned}$$

Now apply Lemma 4.4.5, we have

$$\mathbb{E}^T\Big[(1 + \frac{1}{A_{Y_3}})\Theta_\lambda(Y_2, y, Y_3)\Big] \le c_{25}(q_1 + \delta)^{-|Y_3|+|Y_2|+1}. \tag{4.36}$$

Applying Cauchy-Schwartz inequality to $\mathbb{E}^T\Big[\frac{S_{\lambda,[\![\rho,Y_2]\!]}}{p_1(\rho,Y_2)^2}\Big]$ yields

$$\begin{aligned}
\Delta_1(n) &\le c_{23} \sum_{|y|=n, Y_1=\rho} 2\Bigg(\sqrt{\mathbb{E}^T\Big[S^2_{\lambda,[\![\rho,Y_2]\!]}\Big]\mathbb{E}^T\Big[\frac{1}{p_1(\rho,Y_2)^4}\Big]}\Bigg)\mathbb{E}^T\Big[(1 + \frac{1}{A_{Y_3}})\Theta_\lambda(Y_2, y, Y_3)\Big] \\
&\le c_{26} \sum_{|y|=n, Y_1=\rho} \sqrt{\mathbb{E}^T\Big[\frac{1}{p_1(\rho,Y_2)^4}\Big]}\mathbb{E}^T\Big[(1 + \frac{1}{A_{Y_3}})\Theta_\lambda(Y_2, y, Y_3)\Big],
\end{aligned}$$

where the last inequality holds because $\mathbb{E}^T\Big[S^2_{\lambda,[\![\rho,Y_2]\!]}\Big] \le c_{27}(n_0) < \infty$. By (4.36),

$$\begin{aligned}
\Delta_1(n) &\le c_{28} \sum_{|y|=n, Y_1=\rho} \mathbb{E}^T\Big[\frac{1}{p_1(\rho,Y_2)^4}\Big](q_1 + \delta)^{-|Y_3|+|Y_2|+1} \\
&= c_{28}\mathbb{E}^T\Big[\sum_{|u|=n_0} \mathbb{1}_{Z^T_{n_0} > K_0}\frac{1}{p_1(\rho,u)^4}\Big] \sum_{y:|y|=n, Y_2=u} (q_1 + \delta)^{-|Y_3|+n_0+1}
\end{aligned}$$

Observe that

$$\sum_{y:|y|=n, Y_2=u} (q_1 + \delta)^{-|Y_3|+n_0+1} \le \sum_{z:|z|>n, z\in\mathcal{W}(T_u)} (q_1 + \delta)^{-|z|+n_0+1}.$$

Hence,

$$\Delta_1(n) \le c_{28}\mathbb{E}^T\Big[\sum_{|u|=n_0} \mathbb{1}_{Z^T_{n_0} > K_0}\frac{1}{p_1(\rho,u)^4}\Big] \sum_{z:|z|>n, z\in\mathcal{W}(T_u)} (q_1 + \delta)^{-|z|+n_0+1}.$$

Taking expectation under $GW(dT)$ implies that

$$\mathbf{E}_{\mathbf{Q}}[\Delta_1(n)] \le c_{28}\mathbb{E}\Big[\sum_{|u|=n_0} \mathbb{1}_{Z^T_{n_0}>K_0}\frac{1}{p_1(\rho,u)^4}\Big]\mathbf{E}_{\mathbf{Q}}\Big[\sum_{z:|z|>n-n_0, z\in\mathcal{W}} (q_1+\delta)^{-|z|+1}\Big],$$

which by Lemma 4.4.12 is bounded by

$$c_{29}\mathbf{E}_{\mathbf{Q}}\Big[\sum_{z:|z|>n-n_0, z\in\mathcal{W}} (q_1+\delta)^{-|z|+1}\Big] = c_{29}\sum_{l>n/n_0-1}\mathbf{E}_{\mathbf{Q}}\Big[\sum_{|z|=ln_0, z\in\mathcal{W}} (q_1+\delta)^{-|z|+1}\Big]$$

Recall that $\mathcal{W}$ is a GW tree of mean $\mathbf{E}[Z_{n_0}; Z_{n_0}\le K_0]\le r^{n_0}$. We can choose r to be $q_1+\delta/2$ so that

$$\sum_{l\ge 1}\mathbf{E}_{\mathbf{Q}}\Big[\sum_{|z|=ln_0, z\in\mathcal{W}} (q_1+\delta)^{-|z|+1}\Big] \le \sum_{l\ge 1}(q_1+\delta)^{-ln_0+1}r^{ln_0} < c_{30}\gamma^{l_0},$$

where $\gamma := (\frac{q_1+\delta/2}{q_1+\delta})^{n_0} < 1$ and $l_0 := \lceil \frac{n}{n_0}\rceil - 1 = j-1$. As a result, for any $n>n_0$,

$$\mathbf{E}_{\mathbf{Q}}[\Delta_1(n)] \le c_{31}\gamma^{l_0} < \infty. \tag{4.37}$$

Turn to $\Delta_2(n)$. As $P^{\omega,T}(\tau_{Y_1}<\infty)\le P^{\omega,T}(\tau_{\overleftarrow{Y_1}}<\infty)$, one sees that

$$\Delta_2(n) \le 2c_{23}\sum_{l=1}^{j-1}\sum_{|x|=ln_0-1}\sum_{|y|=n, \overleftarrow{Y_1}=x}\mathbb{E}^T\Big[\frac{P^{\omega,T}(\tau_x<\infty)S_{\lambda,[\![Y_1,Y_2]\!]}}{p_1(Y_1,Y_2)^2}\Big]\mathbb{E}^T\Big[(1+\frac{1}{A_{Y_3}})\Theta_\lambda(Y_2,y,Y_3)\Big],$$

which equals to

$$\sum_{l=1}^{j-1}\sum_{|x|=ln_0-1}\sum_{z:\overleftarrow{z}=x}\mathbb{P}^T(\tau_x<\infty)2c_{23}\sum_{|y|=n, Y_1=z}\mathbb{E}^T\Big[\frac{S_{\lambda,[\![Y_1,Y_2]\!]}}{p_1(Y_1,Y_2)^2}\Big]\mathbb{E}^T\Big[(1+\frac{1}{A_{Y_3}})\Theta_\lambda(Y_2,y,Y_3)\Big],$$

as $P^{\omega,T}(\tau_x<\infty)$ and $\frac{S_{\lambda,[\![Y_1,Y_2]\!]}}{p_1(Y_1,Y_2)^2}$ are independent under $\mathbb{P}^T$.

Note that for all $z\in T$, $2c_{23}\sum_{|y|=n,Y_1=z}\mathbb{E}^T\Big[\frac{S_{\lambda,[\![Y_1,Y_2]\!]}}{p_1(Y_1,Y_2)^2}\Big]\mathbb{E}^T\Big[(1+\frac{1}{A_{Y_3}})\Theta_\lambda(Y_2,y,Y_3)\Big]$ are i.i.d. copies of $\Delta_1(n-|z|)$. Taking expectation yields that

$$\begin{aligned}\mathbf{E}_{\mathbf{Q}}[\Delta_2(n)] &\le \sum_{l=1}^{j-1}\mathbb{E}\Big[\sum_{|x|=ln_0-1}\mathbb{1}_{\tau_x<\infty}(d(x)-1)\Big]\mathbf{E}_{\mathbf{Q}}[\Delta_1(n-ln_0)]\\ &\le bc_{31}\sum_{l=1}^{j-1}\mathbb{E}\Big[\sum_{|x|=ln_0-1}\mathbb{1}_{\tau_x<\infty}\Big]\gamma^{j-l-1},\end{aligned}$$

where the last inequality follows from (4.37). By Lemma 4.4.10, for any $j\ge 2$,

$$\mathbf{E}_{\mathbf{Q}}[\Delta_2(n)] \le c_{32}\sum_{l=1}^{j-1}\gamma^{j-1-l} \le c_{33} < \infty.$$

Plugging the above inequality and (4.37) into (4.35) implies that

$$\mathbb{E}[N^\lambda_{n,2}] \le \mathbf{E}_{\mathbf{Q}}[\Delta_1(n)] + \mathbf{E}_{\mathbf{Q}}[\Delta_2(n)] < \infty.$$

The estimate of $\mathbb{E}[(N^*_{n,2})^\lambda]$ follows from similar arguments. We feel free to omit it. $\square$

4.5 Proofs of one dimensional results

Proof of Lemma 4.4.2. For any $i \geq 1$, let $S_i = -\sum_{j=1}^{i} \log(A_j A_{j-1})$ and define $S_0 = 0$. As $i \mapsto \tilde{P}_i^\omega(\tilde{\tau}_{-1} > \tilde{\tau}_n)$ is the solution to the Dirichlet problem

$$\begin{cases} \varphi(-1) = 0, \ \varphi(n) = 1 \\ \tilde{E}_i^\omega(\varphi(\tilde{\eta}_1)) = \varphi(i) \qquad i \in [\![0, n-1]\!]. \end{cases}$$

It follows that

$$\tilde{P}_i^\omega(\tilde{\tau}_{-1} > \tilde{\tau}_n) = \frac{\sum_{j=0}^{i} \exp(S_j)}{\sum_{j=0}^{n} \exp(S_j)}. \tag{4.38}$$

As a consequence, for any $0 \leq l \leq n$,

$$\begin{aligned}
\tilde{P}_0^\omega(\tilde{\tau}_l < \tilde{\tau}_{-1}) &= \frac{1}{\sum_{j=0}^{l} \exp(S_j)} \geq \frac{\exp(-\max_{0\leq j\leq l} S_j)}{l+1} \\
\tilde{P}_{l+1}^\omega(\tilde{\tau}_n < \tilde{\tau}_l) &= \frac{\exp(S_{l+1})}{\sum_{j=l+1}^{n} \exp(S_j)} \leq \exp(-\max_{l+1\leq j\leq n}(S_j - S_{l+1})) \\
\tilde{P}_{l-1}^\omega(\tilde{\tau}_{-1} < \tilde{\tau}_l) &= \frac{\exp(S_l)}{\sum_{j=0}^{l} \exp(S_j)} \leq \exp(-\max_{0\leq j\leq l}(S_j - S_l)).
\end{aligned}$$

We only need to consider n large, take $l = \lfloor z_1 n \rfloor$, note that

$$\begin{aligned}
\tilde{P}_l^\omega(\tilde{\tau}_l^* > \tilde{\tau}_{-1} \wedge \tilde{\tau}_n) &= p(l, l+1)\tilde{P}_{l+1}^\omega(\tilde{\tau}_n < \tilde{\tau}_l) + p(l, l-1)\tilde{P}_{l-1}^\omega(\tilde{\tau}_{-1} < \tilde{\tau}_l) \\
&\leq \max(\tilde{P}_{l+1}^\omega(\tilde{\tau}_n < \tilde{\tau}_l), \tilde{P}_{l-1}^\omega(\tilde{\tau}_{-1} < \tilde{\tau}_l)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\tilde{P}_0^\omega(\tilde{\tau}_n \wedge \tilde{\tau}_{-1} > m) &\geq \tilde{P}_0^\omega(\tilde{\tau}_l < \tilde{\tau}_{-1})\tilde{P}_l^\omega(\tilde{\tau}_l^* < \tilde{\tau}_{-1} \wedge \tilde{\tau}_n)^m \\
&\geq \frac{\exp(-\max_{0\leq j\leq l} S_j)}{l+1} \left(1 - \tilde{P}_l^\omega(\tilde{\tau}_l^* \geq \tilde{\tau}_{-1} \wedge \tilde{\tau}_n)\right)^m \\
&\geq \frac{\exp(-\max_{0\leq j\leq l} S_j)}{l+1} \left(1 - \exp(-\max_{l+1\leq k\leq n}(S_k - S_{l+1}) \wedge \max_{0\leq k\leq l}(S_k - S_l))\right)^m \\
&\geq \frac{\mathbb{1}_{\max_{0\leq k\leq l} S_k \leq 0}}{l+1}(1 - e^{-zn})^m \mathbb{1}_{\max_{l+1\leq k\leq n}(S_k - S_{l+1}) \geq zn} \mathbb{1}_{\max_{0\leq k\leq l}(S_k - S_l) \geq zn}.
\end{aligned}$$

As $m \approx e^{zn}$, we have $(1 - e^{-zn})^m = O(1)$, taking expectation under $\mathbf{P}(\cdot | A_0 \in [a, \frac{1}{a}])$ yields

$$\begin{aligned}
&\tilde{\mathbb{P}}_0(\tilde{\tau}_n \wedge \tilde{\tau}_{-1} > m | A_0 \in [a, \frac{1}{a}]) \\
&\geq \frac{c}{n}\mathbf{P}(\max_{0\leq k\leq l} S_k \leq 0, \ \max_{0\leq k\leq l}(S_k - S_l) \geq zn | A_0 \in [a, \frac{1}{a}])\mathbf{P}(\max_{l+1\leq k\leq n}(S_k - S_{l+1}) \geq zn) \\
&\geq \frac{c}{n}\mathbf{P}(\max_{0\leq k\leq l} S_k \leq 0, \ S_l \leq -zn | A_0 \in [a, \frac{1}{a}])\mathbf{P}((S_n - S_{l+1}) \geq zn).
\end{aligned}$$

For $k \geq 1$, write $\mathcal{S}_k = -\sum_{i=1}^k \log A_i$, then as $S_k = -\log A_0 + \mathcal{S}_{k-1} + \mathcal{S}_k$,

$$\begin{aligned}
&\mathbf{P}(\max_{0\leq k\leq l} S_k \leq 0,\ S_l \leq -zn | A_0 \in [a, \frac{1}{a}]) \\
&\geq \mathbf{P}(A_0 \geq 1, A_l \geq 1, \max_{1\leq k\leq l-1} \mathcal{S}_k \leq 0,\ \mathcal{S}_{l-1} \leq -\frac{zn}{2} | A_0 \in [a, \frac{1}{a}]) \\
&= \mathbf{P}(A_0 \geq 1 | A_0 \in [a, \frac{1}{a}])\mathbf{P}(A_l \geq 1)\mathbf{P}(\max_{1\leq k\leq l-1} \mathcal{S}_k \leq 0,\ \mathcal{S}_{l-1} \leq -\frac{zn}{2})
\end{aligned}$$

note that

$$\mathbf{P}(\max_{1\leq k\leq l-1} \mathcal{S}_k \leq 0,\ \mathcal{S}_{l-1} \leq -\frac{zn}{2}) \geq \frac{1}{l}\mathbf{P}(\mathcal{S}_{l-1} \leq -\frac{zn}{2})$$

and

$$S_n - S_{l+1} = -\log A_{l+1} - \log A_n - 2\sum_{k=l+2}^{n-1} \log A_k.$$

Therefore,

$$\begin{aligned}
\tilde{\mathbb{P}}_0(\tilde{\tau}_n \wedge \tilde{\tau}_{-1} > m | A_0 \in [a, \frac{1}{a}]) &\geq \frac{c}{n^2}\mathbf{P}(\mathcal{S}_{l-1} \leq -\frac{zn}{2})\mathbf{P}(S_n - S_{l+1} \geq zn) \\
&\geq \frac{c}{n^2}\mathbf{P}(\mathcal{S}_{l-1} \leq -\frac{zn}{2})\mathbf{P}(A_{l+1} \leq 1)\mathbf{P}(A_n \leq 1)\mathbf{P}(-\sum_{k=l+2}^{n-1} \log A_k \geq \frac{zn}{2}) \\
&\geq \frac{c}{n^2}\mathbf{P}(\mathcal{S}_{l-1} \leq -\frac{zn}{2})\mathbf{P}(-\sum_{k=l+2}^{n-1} \log A_k \geq \frac{zn}{2}) \\
&\geq \frac{c}{n^2}\mathbf{P}(\sum_{k=1}^{l-1} \log A_k \geq \frac{zn}{2})\mathbf{P}(\sum_{k=l+2}^{n-1} \log A_k \leq -\frac{zn}{2})
\end{aligned}$$

Applying Cramér's theorem to sums of i.i.d. random variables $\log A_k$, we have

$$\tilde{\mathbb{P}}_0(\tilde{\tau}_n \wedge \tilde{\tau}_{-1} > m | A_0 \in [a, \frac{1}{a}]) \gtrsim \exp(-n\left(z_1 I(\frac{z}{2z_1}) + (1-z_1)I(\frac{-z}{2(1-z_1)})\right))$$

where $I(x) = \sup_{t\in\mathbb{R}}\{tx - \log \mathbf{E}(A^t)\}$ is the associated rate function. □

Proof of Lemma 4.4.3. Replace $I(\frac{-z}{2(1-z_1)})$ using

$$\begin{aligned}
I(-x) &= \sup_{t\in\mathbb{R}}\{-tx - \log \mathbf{E}(A^t)\} = \sup_{t\in\mathbb{R}}\{-tx - \log \mathbf{E}(A^{1-t})\} \\
&= \sup_{s\in\mathbb{R}}\{-(1-s)x - \log \mathbf{E}(A^s)\} = I(x) - x.
\end{aligned}$$

For fixed z, by convexity of the rate function I, the supremum of $-z_1 I(\frac{z}{2z_1}) - (1-z_1)I(\frac{z}{2(1-z_1)})$ is obtained when $z_1 = \frac{1}{2}$, we are left to compute

$$\sup_{0<z}\{\frac{\log q_1 - I(z)}{z} + \frac{1}{2}\},$$

clearly, $\frac{\log q_1 - I(z)}{z} \leq -t^*$, when z is such that $(t \mapsto \log \mathbf{E}(A^t))'(t^*) = z > 0$, the maximum is obtained. □

Proof of Lemma 4.4.4 . Observe that

$$\widetilde{P}^{\omega}_{Y_1}(\tilde{\tau}_y < \tilde{\tau}_{\overleftarrow{Y_1}})\widetilde{G}^{\tilde{\tau}_{Y_1}\wedge\tilde{\tau}_{Y_3}}(y,y) = \widetilde{P}^{\omega}_{Y_1}(\tilde{\tau}_y < \tilde{\tau}_{\overleftarrow{Y_1}} \wedge \tilde{\tau}_{Y_3})\widetilde{E}^{\omega}_y\left[\sum_{k=0}^{\tilde{\tau}_{Y_1}\wedge\tilde{\tau}_{Y_3}} 1_{\{\widetilde{\eta}_k=y\}}\right]$$

$$\leq \widetilde{P}^{\omega}_{Y_1}(\tilde{\tau}_y < \tilde{\tau}_{\overleftarrow{Y_1}} \wedge \tilde{\tau}_{Y_3})\widetilde{E}^{\omega}_y\left[\sum_{k=0}^{\tilde{\tau}_{\overleftarrow{Y_1}}\wedge\tilde{\tau}_{Y_3}} 1_{\{\widetilde{\eta}_k=y\}}\right] = \widetilde{E}^{\omega}_{Y_1}\left[\sum_{k=0}^{\tilde{\tau}_{\overleftarrow{Y_1}}\wedge\tilde{\tau}_{Y_3}} 1_{\{\widetilde{\eta}_k=y\}}\right].$$

Obviously,

$$\widetilde{E}^{\omega}_{Y_1}\left[\sum_{k=0}^{\tilde{\tau}_{\overleftarrow{Y_1}}\wedge\tilde{\tau}_{Y_3}} 1_{\{\widetilde{\eta}_k=y\}}\right] \leq \widetilde{E}^{\omega}_{Y_1}[\tilde{\tau}_{\overleftarrow{Y_1}} \wedge \tilde{\tau}_{Y_3}].$$

This gives us (4.11).

Moreover, to get (4.12), we only need to show that for any $0 \leq p < m$, we have

$$\widetilde{E}^{\omega}_p[\tilde{\tau}_{p-1} \wedge \tilde{\tau}_m] \leq 1 + A_pA_{p+1} + A_pA_{p+1}\widetilde{E}^{\omega}_{p+1}[\tilde{\tau}_p \wedge \tilde{\tau}_m]. \tag{4.39}$$

In fact, since $0 \leq \lambda \leq 1$, (4.39) implies that

$$\widetilde{E}^{\omega}_p[\tilde{\tau}_{p-1} \wedge \tilde{\tau}_m]^{\lambda} \leq 1 + (A_pA_{p+1})^{\lambda} + (A_pA_{p+1})^{\lambda}\widetilde{E}^{\omega}_{p+1}[\tilde{\tau}_p \wedge \tilde{\tau}_m]^{\lambda}.$$

applying this inequality a few times along the interval $[\![Y_1, Y_3]\!]$, we obtain (4.12). It remains to show (4.39). Observe that

$$\begin{aligned}
&\widetilde{E}^{\omega}_p[\tilde{\tau}_{p-1} \wedge \tilde{\tau}_m] = \widetilde{\omega}(p,p-1) + \widetilde{\omega}(p,p+1)(1 + \widetilde{E}^{\omega}_{p+1}[\tilde{\tau}_{p-1} \wedge \tilde{\tau}_m])\\
&= 1 + \widetilde{\omega}(p,p+1)\widetilde{E}^{\omega}_{p+1}[\tilde{\tau}_{p-1} \wedge \tilde{\tau}_m]\\
&= 1 + \widetilde{\omega}(p,p+1)\Big(\widetilde{E}^{\omega}_{p+1}[\tilde{\tau}_m; \tilde{\tau}_m < \tilde{\tau}_p] + \widetilde{E}^{\omega}_{p+1}[\tilde{\tau}_p; \tilde{\tau}_p < \tilde{\tau}_m] + \widetilde{P}^{\omega}_{p+1}(\tilde{\tau}_p < \tilde{\tau}_m)\widetilde{E}^{\omega}_p[\tilde{\tau}_{p-1} \wedge \tilde{\tau}_m]\Big).
\end{aligned}$$

It follows that

$$\begin{aligned}
\widetilde{E}^{\omega}_p[\tilde{\tau}_{p-1} \wedge \tilde{\tau}_m] &= \frac{1 + \widetilde{\omega}(p,p+1)\widetilde{E}^{\omega}_{p+1}[\tilde{\tau}_p \wedge \tilde{\tau}_m]}{1 - \widetilde{\omega}(p,p+1)\widetilde{P}^{\omega}_{p+1}(\tilde{\tau}_p < \tilde{\tau}_m)}\\
&= \frac{1 + \widetilde{\omega}(p,p+1)\widetilde{E}^{\omega}_{p+1}[\tilde{\tau}_p \wedge \tilde{\tau}_m]}{\widetilde{\omega}(p,p-1) + \widetilde{\omega}(p,p+1)\widetilde{P}^{\omega}_{p+1}(\tilde{\tau}_m < \tilde{\tau}_p)} \leq \frac{1 + \widetilde{\omega}(p,p+1)\widetilde{E}^{\omega}_{p+1}[\tilde{\tau}_p \wedge \tilde{\tau}_m]}{\widetilde{\omega}(p,p-1)}.
\end{aligned}$$

Therefore,

$$\widetilde{E}^{\omega}_p[\tilde{\tau}_{p-1} \wedge \tilde{\tau}_m] \leq (1 + A_pA_{p+1}) + A_pA_{p+1}\widetilde{E}^{\omega}_{p+1}[\tilde{\tau}_p \wedge \tilde{\tau}_m].$$

□

Proof of Lemma 4.4.5. Recall that $\mathbf{E}[A^t] < \infty$ for any $t \in \mathbb{R}$. By Hölder's inequality, it suffices to show that there exists some $\delta' > 0$ such that for all n large enough,

$$\mathbf{E}\left[\left(\tilde{E}^{\omega}_0[\tilde{\tau}_{-1} \wedge \tilde{\tau}_n]\right)^{\lambda(1+\delta')}\right] \leq (q_1 + \delta)^{-n}. \tag{4.40}$$

It remains to prove (4.40). In fact, we only need to show that for $1 > \lambda' = \lambda(1+\delta) > 0$,

$$\limsup_{n\to\infty} \frac{\log \mathbf{E}\left[\left(\tilde{E}^{\omega}_0[\tilde{\tau}_{-1} \wedge \tilde{\tau}_n]\right)^{\lambda'}\right]}{n} \leq \psi(\lambda' + 1/2) \tag{4.41}$$

where $\psi(t) = \log \mathbf{E}(A^t)$. One therefore sees that if $t^* - 1/2 > \lambda'$, then $\psi(\lambda' + 1/2) < \psi(t^*) = -\log q_1$. To show (4.41), recall that for any $0 \le i \le n-1$,

$$\begin{aligned}\widetilde{G}^{\tilde\tau_{-1}\wedge\tilde\tau_n}(i,i) &= \widetilde{E}_i^{\omega}\Big[\sum_{k=0}^{\tilde\tau_{-1}\wedge\tilde\tau_n} 1_{\eta=i}\Big]\\ &= \frac{1}{1-\widetilde{\omega}(i,i-1)\widetilde{P}_{i-1}(\tilde\tau_i<\tilde\tau_{-1})-\widetilde{\omega}(i,i+1)\widetilde{P}_{i+1}(\tilde\tau_i<\tilde\tau_n)}.\end{aligned}$$

Then, $\widetilde{E}_0^{\omega}[\tilde\tau_{-1}\wedge\tilde\tau_n] = 1 + \sum_{i=0}^{n-1}\widetilde{P}_0^{\omega}(\tilde\tau_i<\tilde\tau_{-1})\widetilde{G}^{\tilde\tau_{-1}\wedge\tilde\tau_n}(i,i)$ implies that

$$\widetilde{E}_0^{\omega}[\tilde\tau_{-1}\wedge\tilde\tau_n] = 1 + \sum_{i=0}^{n-1}\frac{\widetilde{P}_0^{\omega}(\tilde\tau_i<\tilde\tau_{-1})}{\widetilde{\omega}(i,i-1)\widetilde{P}_{i-1}^{\omega}(\tilde\tau_{-1}<\tilde\tau_i)+\widetilde{\omega}(i,i+1)\widetilde{P}_{i+1}^{\omega}(\tilde\tau_n<\tilde\tau_i)}.$$

Recall that by (4.38), if $S_i := \sum_{j=1}^{i} -\log(A_{j-1}A_j)$ for $i\ge 1$ and $S_0 = 0$, then

$$\begin{aligned}\widetilde{P}_0^{\omega}(\tilde\tau_i<\tilde\tau_{-1}) &= \frac{1}{\sum_{k=0}^{i} e^{S_k}}\\ \widetilde{P}_{i-1}^{\omega}(\tilde\tau_{-1}<\tilde\tau_i) &= \frac{e^{S_i}}{\sum_{k=0}^{i} e^{S_k}}\\ \widetilde{P}_{i+1}^{\omega}(\tilde\tau_n<\tilde\tau_i) &= \frac{1}{\sum_{k=i+1}^{n} e^{S_k-S_{i+1}}}\end{aligned}$$

It is immediate that

$$\begin{aligned}\frac{\widetilde{P}_0^{\omega}(\tilde\tau_i<\tilde\tau_{-1})}{\widetilde{\omega}(i,i-1)\widetilde{P}_{i-1}^{\omega}(\tilde\tau_{-1}<\tilde\tau_i)+\widetilde{\omega}(i,i+1)\widetilde{P}_{i+1}^{\omega}(\tilde\tau_n<\tilde\tau_i)} &= \frac{\frac{1}{\sum_{k=0}^{i}e^{S_k}}}{\frac{1}{1+A_iA_{i+1}}\frac{e^{S_i}}{\sum_{k=0}^{i}e^{S_k}}+\frac{A_iA_{i+1}}{1+A_iA_{i+1}}\frac{1}{\sum_{k=i+1}^{n}e^{S_k-S_{i+1}}}}\\ &\le \frac{1}{\frac{1}{1+A_iA_{i+1}}\frac{e^{S_i}}{\sum_{k=0}^{i}e^{S_k}}+\frac{A_iA_{i+1}}{1+A_iA_{i+1}}\frac{1}{\sum_{k=i+1}^{n}e^{S_k-S_{i+1}}}}.\end{aligned}$$

Let $X_k = -\log A_k$. For any $0\le i\le n$, define

$$\begin{aligned}H_i(-X) &:= \max_{0\le j\le i}(-X_j - X_{j+1} - \cdots - X_{i-1})\\ H_{n-i-1}(X) &:= \max_{i+2\le j\le n}(X_{i+2}+\cdots+X_j)\end{aligned}$$

Note that

$$S_k - S_i \le 2H_i(-X) + (-X_i)_+, \forall 0\le k\le i,$$

and that

$$S_k - S_{i+1} \le 2H_{n-i-1}(X) + (X_{i+1})_+, \forall i+1\le k\le n.$$

Then,

$$\frac{1}{1+A_iA_{i+1}}\frac{e^{S_i}}{\sum_{k=0}^{i}e^{S_k}} \ge \frac{1}{1+A_iA_{i+1}}\frac{1}{(1+i)e^{2H_i(-X)+(-X_i)_+}} \ge \frac{1}{n(A_i+1)(1+A_iA_{i+1})}e^{-2H_i(-X)}.$$

Similarly,

$$\frac{A_iA_{i+1}}{1+A_iA_{i+1}}\frac{1}{\sum_{k=i+1}^{n}e^{S_k-S_{i+1}}} \ge \frac{(A_{i+1}\wedge 1)A_iA_{i+1}}{n(1+A_iA_{i+1})}e^{-2H_{n-i-1}(X)}.$$

So,

$$
\begin{aligned}
&\frac{1}{1+A_iA_{i+1}}\frac{e^{S_i}}{\sum_{k=0}^{i}e^{S_k}}+\frac{A_iA_{i+1}}{1+A_iA_{i+1}}\frac{1}{\sum_{k=i+1}^{n}e^{S_k-S_{i+1}}}\\
&\qquad\geq\frac{1}{n(A_i+1)(1+A_iA_{i+1})}e^{-2H_i(-X)}+\frac{(A_{i+1}\wedge 1)A_iA_{i+1}}{n(1+A_iA_{i+1})}e^{-2H_{n-i-1}(X)}\\
&\qquad\qquad\geq\frac{1}{n}\Big(\frac{1}{(A_i\vee 1)(1+A_iA_{i+1})}\wedge\frac{(A_{i+1}\wedge 1)A_iA_{i+1}}{1+A_iA_{i+1}}\Big)e^{-2H_i(-X)}\vee e^{-2H_{n-i-1}(X)}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\frac{\widetilde{P}_0^{\omega}(\tilde{\tau}_i<\tilde{\tau}_{-1})}{\widetilde{\omega}(i,i-1)\widetilde{P}_{i-1}^{\omega}(\tilde{\tau}_{-1}<\tilde{\tau}_i)+\widetilde{\omega}(i,i+1)\widetilde{P}_{i+1}^{\omega}(\tilde{\tau}_n<\tilde{\tau}_i)}\\
&\leq n\Big((A_i\vee 1)(1+A_iA_{i+1})+\frac{1+A_iA_{i+1}}{(A_{i+1}\wedge 1)A_iA_{i+1}}\Big)e^{2H_i(-X)\wedge H_{n-i-1}(X)}.
\end{aligned}
$$

Thus, for any $\lambda\leq 1$, $n\geq 2$,

$$
\widetilde{E}_0^{\omega}[\tilde{\tau}_{-1}\wedge\tilde{\tau}_n]^{\lambda}\lesssim n+n^2\sum_{i=0}^{n-1}\Big((A_i\vee 1)(1+A_iA_{i+1})+\frac{1+A_iA_{i+1}}{(A_{i+1}\wedge 1)A_iA_{i+1}}\Big)^{\lambda}e^{2\lambda H_i(-X)\wedge H_{n-i-1}(X)}
$$

By independence,

$$
\mathbf{E}\widetilde{E}_0^{\omega}[\tilde{\tau}_{-1}\wedge\tilde{\tau}_n]^{\lambda}\lesssim n+n^3\max_{0\leq i\leq n-1}\mathbf{E}[e^{2\lambda H_i(-X)\wedge H_{n-i-1}(X)}] \tag{4.42}
$$

Recall that $\psi(\lambda)=\log\mathbf{E}[A^{\lambda}]$ and $\mathcal{S}_k=-\sum_{i=1}^{k}\log A_i$. Let $t>0$, for $i\geq 1$, $x>0$,

$$
\begin{aligned}
\mathbf{P}(H_i(-X)\geq xi)&\leq\mathbf{P}(\max_{0\leq k\leq i}[-t\mathcal{S}_k-\psi(t)k]\geq xti-\psi(t)i)\\
&\leq\mathbf{P}(\max_{0\leq k\leq i}e^{-t\mathcal{S}_k-\psi(t)k}\geq e^{(xt-\psi(t))i})\\
&\leq e^{-(xt-\psi(t))i},
\end{aligned} \tag{4.43}
$$

where the last inequality stem from Doob's maximal inequality and the fact that $(e^{-t\mathcal{S}_j-\psi(t)j})_j$ is a martingale. Since $x\geq\mathbf{E}(\log A)$, $I(x)=\sup_{t>0}\{tx-\psi(t)\}$, we have

$$
\mathbf{P}(H_i(-X)\geq xi)\leq e^{-I(x)i}. \tag{4.44}
$$

Similarly, for any $j\geq 1$ and $x>\mathbf{E}[-\log A]$.

$$
\begin{aligned}
\mathbf{P}(H_j(X)\geq xj)&\leq\mathbf{P}(\max_{0\leq k\leq j}[t\mathcal{S}_k-\psi(-t)k]\geq xtj-\psi(-t)j)\\
&\leq\mathbf{P}(\max_{0\leq k\leq j}e^{t\mathcal{S}_k-\psi(-t)k}\geq e^{(xt-\psi(-t))j})\\
&\leq e^{-(xt-\psi(-t))j},
\end{aligned} \tag{4.45}
$$

which implies that

$$
\mathbf{P}(H_j(X)\geq xj)\leq e^{-I(-x)j}. \tag{4.46}
$$

Further, for $0<x<\mathbf{E}[-\log A]$, one sees that by Cramér's theorem,

$$
\begin{aligned}
\mathbf{P}(H_j(X)\leq xj)&\leq\mathbf{P}(X_1+\cdots+X_j\leq xj)\\
&=\mathbf{P}(-X_1-\cdots-X_j\geq -xj)\leq e^{-I(-x)j}.
\end{aligned} \tag{4.47}
$$

Take $\eta > 0$. In (4.42), we can replace $H_i(-X) \wedge H_{n-i-1}(X)$ by $H_i(-X) \wedge H_{n-i-1}(X) \wedge K\eta n$ with some $K \geq 1$ large enough. In fact,

$$\mathbf{E}[e^{2\lambda H_i(-X)\wedge H_{n-i-1}(X)}] \leq \underbrace{\mathbf{E}[e^{2\lambda H_i(-X)\wedge H_{n-i-1}(X)}; H_i(-X) \vee H_{n-i-1}(X) \leq K\eta n]}_{\Xi_K^-(i)}$$
$$+ \underbrace{\mathbf{E}[e^{2\lambda H_i(-X)\wedge H_{n-i-1}(X)}; H_i(-X) \vee H_{n-i-1}(X) \geq K\eta n]}_{=:\Xi_K^+(i)}.$$

Observe that

$$\begin{aligned}\Xi_K^+(i) &\leq \mathbf{E}[e^{2\lambda H_i(-X)}; H_i(-X) \geq K\eta n] + \mathbf{E}[e^{2\lambda H_{n-i-1}(X)}; H_{n-i-1}(X) \geq K\eta n]\\ &=: \Xi_1 + \Xi_2\end{aligned}$$

Let us bound Ξ_1,

$$\begin{aligned}\Xi_1 =&\mathbf{E}\int_{-\infty}^{H_i(-X)} 2\lambda e^{2\lambda x}\mathbf{1}_{H_i(-X)\geq K\eta n}dx = \int_{\mathbb{R}} 2\lambda e^{2\lambda x}\mathbf{P}(H_i(-X) \geq K\eta n \vee x)dx\\ =&\int_{-\infty}^{K\eta n} 2\lambda e^{2\lambda x}dx\mathbf{P}(H_i(-X) \geq K\eta n) + \int_{K\eta n}^{\infty} 2\lambda e^{2\lambda x}\mathbf{P}(H_i(-X) \geq x)dx\\ =&e^{2\lambda K\eta n}\mathbf{P}(H_i(-X) \geq K\eta n) + \int_K^{\infty} 2\lambda\eta n e^{2\lambda t\eta n}\mathbf{P}(H_i(-X) \geq t\eta n)dt\end{aligned}$$

By applying (4.43), one sees that for any $0 \leq i \leq n-1$ and $\mu = 3 > 2\lambda$,

$$\begin{aligned}\Xi_1 \leq&e^{2\lambda K\eta n}e^{-\mu K\eta n+\psi(\mu)i} + \int_K^{\infty} 2\lambda\eta n e^{2\lambda t\eta n}e^{-\mu t\eta n+\psi(\mu)i}dt\\ \leq&e^{-K\eta n+\psi(3)n} + 2\lambda e^{\psi(3)n}\int_K^{\infty} \eta n e^{-t\eta n}dt\\ \leq&3e^{-K\eta n+\psi(3)n},\end{aligned}$$

which is less than 1 when we choose K large enough. Similarly, we can show that for any $i \leq n-1$,

$$\Xi_2 \leq 1,$$

for K large enough. Consequently, (4.42) becomes that

$$\mathbf{E}\widetilde{E}_0^{\omega}[\tilde{\tau}_{-1} \wedge \tilde{\tau}_n]^{\lambda} \lesssim 3n^3 + n^3 \max_{0\leq i\leq n-1} \Xi_K^-(i). \tag{4.48}$$

It remains to bound $\Xi_K^-(i)$. Take sufficiently small $\varepsilon > 0$ and let $L = \lfloor \frac{1}{\varepsilon} \rfloor$. For any i such that $l_1\lfloor \varepsilon n\rfloor \leq i < (l_1+1)\lfloor \varepsilon n\rfloor$ and $l_2\lfloor \varepsilon n\rfloor \leq n-i-1 < (l_2+1)\lfloor \varepsilon n\rfloor$ with $0 \leq l_1, l_2 \leq L$, we have

$$\begin{aligned}\Xi_K^-(i) \leq& \sum_{0\leq k_1,k_2\leq K} e^{2\lambda k_1\wedge k_2\eta n+2\lambda\eta n}\mathbf{P}(k_1\eta n \leq H_i(-X) < (k_1+1)\eta n)\mathbf{P}(k_2\eta n \leq H_{n-i-1}(X) < (k_2+1)\eta n)\\ \leq& \sum_{0\leq k_1,k_2\leq K} e^{2\lambda k_1\wedge k_2\eta n+2\lambda\eta n}\mathbf{P}(H_i(-X) \geq k_1\eta n)\mathbf{P}(k_2\eta n \leq H_{n-i-1}(X) < (k_2+1)\eta n).\end{aligned}$$

By (4.44), we have

$$\mathbf{P}(H_i(-X) \geq k_1\eta n) \leq e^{-I(x_1)i}$$

where x_1 is the point in $[\frac{k_1\eta n}{(l_1+1)\lfloor\varepsilon n\rfloor}, \frac{k_1\eta n}{l_1\lfloor\varepsilon n\rfloor}]$ where I reaches the minimum in this interval. By large deviation estimates (4.46) (4.47), we have

$$\mathbf{P}(k_2\eta n \le H_{n-i-1}(X) < (k_2+1)\eta n) \le e^{-I(x_2)(n-i)}$$

where x_2 is the point in $[\frac{k_1\eta n}{(l_2+1)\lfloor\varepsilon n\rfloor}, \frac{(k_2+1)\eta n}{l_2\lfloor\varepsilon n\rfloor}]$ where I reaches the minimum in this interval. Therefore,

$$\Xi_K^-(i) \le \sum_{0\le k_1,k_2\le K} e^{2\lambda k_1\wedge k_2\eta n + 2\lambda\eta n} e^{-I(x_1)l_1\lfloor\varepsilon n\rfloor} e^{-I(-x_2)l_2\lfloor\varepsilon n\rfloor}$$

Taking maximum over all l_1, l_2, k_1, k_2 yields that

$$\mathbf{E}\widetilde{E}_0^{\omega}[\tilde{\tau}_{-1}\wedge\tilde{\tau}_n]^{\lambda} \lesssim 3n^2 + n^2K^2 \max_{l_1,l_2,k_1,k_2} \exp\{2\lambda k_1\wedge k_2\eta n + 2\lambda\eta n - I(x_1)l_1\lfloor\varepsilon n\rfloor - I(-x_2)l_2\lfloor\varepsilon n\rfloor\}. \quad (4.49)$$

Observe that

$$\begin{aligned}&2\lambda k_1\wedge k_2\eta n + 2\lambda\eta n - I(x_1)l_1\lfloor\varepsilon n\rfloor - I(-x_2)l_2\lfloor\varepsilon n\rfloor\\ \le &2\lambda(x_1l_1\wedge x_2l_2)\lfloor\varepsilon n\rfloor - I(x_1)l_1\lfloor\varepsilon n\rfloor - I(-x_2)l_2\lfloor\varepsilon n\rfloor + 3\lambda\eta n.\end{aligned}$$

Define

$$L(\lambda) := \sup_{\mathcal{D}}\{\left(x_1z_1\wedge x_2z_2\right)\lambda - I(x_1)z_1 - I(-x_2)z_2\},$$

where $\mathcal{D} := \{x_1, x_2, z_1, z_2 \ge 0, z_1+z_2\le 1\}$.

By Lemma 8.1 in [1], one concludes that

$$\limsup_{n\to\infty}\frac{\log\mathbf{E}\widetilde{E}_0^{\omega}[\tilde{\tau}_{-1}\wedge\tilde{\tau}_n]^{\lambda}}{n} \le L(2\lambda) = \psi(\frac{1+2\lambda}{2}).$$

□

4.6 Some observations on random walks on random trees

Proof of Lemma 4.4.10. As $\beta(x)$ is identically distributed under $\mathbb{P}$,

$$\begin{aligned}\mathbb{E}_{\rho}(\sum_{|x|=n}\mathbb{1}_{\tau_x<\infty})\mathbb{E}(\beta) &= \mathbb{E}[\sum_{|x|=n}P_{\rho}^{\omega,T}(\tau_x<\infty)]\mathbb{E}(\beta)\\ &= \mathbb{E}\left(\sum_{|x|=n}\mathbb{E}^T(P_{\rho}^{\omega,T}(\tau_x<\infty))\mathbb{E}^T(\beta(x))\right).\end{aligned}$$

$P_{\rho}^{\omega,T}(\tau_x<\infty)$ is an increasing function of A_x since

$$\begin{aligned}P_{\rho}^{\omega,T}(\tau_x<\infty) &= P_{\rho}^{\omega,T}(\tau_{\overleftarrow{x}}<\infty)\left(\sum_{k\ge0}P_{\overleftarrow{x}}^{\omega,T}(\tau_{\overleftarrow{x}}^*<\min(\tau_x,\infty))^k\right)p(\overleftarrow{x},x)\\ &= \frac{P_{\rho}^{\omega,T}(\tau_{\overleftarrow{x}}<\infty)}{1-P_{\overleftarrow{x}}^{\omega,T}(\tau_{\overleftarrow{x}}^*<\min(\tau_x,\infty))}\frac{A_{\overleftarrow{x}}A_x}{1+A_{\overleftarrow{x}}B_{\overleftarrow{x}}},\end{aligned}$$

recall that $\beta(x)$ is also an increasing function of A_x, moreover, conditionally on A_x, $P_\rho^{\omega,T}(\tau_x < \infty)$ and $\beta(x)$ are independent, thus by FKG inequality,

$$\begin{aligned}\mathbb{E}^T(P_\rho^{\omega,T}(\tau_x < \infty)\beta(x)) &= \mathbb{E}^T(\mathbb{E}^T(P_\rho^{\omega,T}(\tau_x < \infty)\beta(x)|A_x)) \\ &= \mathbb{E}^T(\mathbb{E}^T(P_\rho^{\omega,T}(\tau_x < \infty)|A_x)\mathbb{E}^T(\beta(x)|A_x)) \\ &\geq \mathbb{E}^T(P_\rho^{\omega,T}(\tau_x < \infty))\mathbb{E}^T(\beta(x))\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}\left(\sum_{|x|=n}\mathbb{E}^T(P_\rho^{\omega,T}(\tau_x < \infty))\mathbb{E}^T(\beta(x))\right) &\leq \mathbb{E}\left(\sum_{|x|=n}\mathbb{E}^T(P_\rho^{\omega,T}(\tau_x < \infty)\beta(x))\right) \\ &= \mathbb{E}\left(\sum_{|x|=n}P_\rho^{\omega,T}(\tau_x < \infty)\beta(x)\right)\end{aligned}$$

For any GW tree and any trajectory on the tree, there is at most one regeneration time at the n-th generation, therefore,

$$\sum_{|x|=n}\mathbb{1}_{\tau_x<\infty,\ \eta_k\neq\overleftarrow{x},\forall k>\tau_x} \leq 1$$

By taking expectation w.r.t. $E_\rho^{\omega,T}$ and using the Markov property at τ_x,

$$\sum_{|x|=n}P_\rho^{\omega,T}(\tau_x < \infty)\beta(x) \leq 1$$

Whence

$$\mathbb{E}(\sum_{|x|=n}\mathbb{1}_{\tau_x<\infty})\mathbb{E}(\beta) \leq 1$$

By transient assumption it suffices to take $c_{11} = \frac{1}{\mathbb{E}(\beta)} < \infty$. □

Proof of Lemma 4.4.11 and Corollary 4.4.1. Let T_i, $i \geq 1$ be independent copies of GW tree with offspring distribution (q), each endowed with independent environment $(\omega_x, x \in T_i)$. Let $\rho^{(i)}$ be the root of T_i. In such setting, $\beta(\rho^{(i)})$, $i \geq 1$ are i.i.d. sequence with common distribution β.

For each T_i, take the left most infinite ray, denoted $v_0^{(i)} = \rho^{(i)}, v_1^{(i)}, \ldots, v_n^{(i)}, \ldots$ Let $\Omega(x) = \{y \neq x;\ \overleftarrow{x} = \overleftarrow{y}\}$ be the set of all brothers of x. Fix some constant C, define

$$R_i = \inf\{n \geq 1;\ \exists z \in \Omega(v_n^{(i)}),\ \frac{1}{A_z\beta(z)} \leq C\}.$$

By Equation (4.15),

$$\frac{1}{\beta(v_{R_i-1}^{(i)})} \leq 1 + \frac{1}{A_{v_{R_i-1}^{(i)}}A_z\beta(z)} \leq 1 + \frac{C}{A_{v_{R_i-1}^{(i)}}}.$$

Also R_i and $\{A_{v_n^{(i)}},\ n \geq 0\}$ are independent under Q. By iteration,

$$\begin{aligned}\frac{1}{\beta(\rho^{(i)})} &\leq 1 + \frac{1}{A_{v_0^{(i)}}A_{v_1^{(i)}}\beta(v_1^{(i)})} \leq 1 + \frac{1}{A_{v_0^{(i)}}A_{v_1^{(i)}}}(1 + \frac{1}{A_{v_1^{(i)}}A_{v_2^{(i)}}\beta(v_2^{(i)})}) \\ &\leq \cdots \\ &\leq 1 + \sum_{k=1}^{R_i-1}\frac{1}{A_{v_0^{(i)}}A_{v_k^{(i)}}}\prod_{j=1}^{k-1}A_{v_j^{(i)}}^{-2} + \frac{C}{A_{v_0^{(i)}}}\prod_{l=1}^{R_i-1}A_{v_l^{(i)}}^{-2}.\end{aligned}$$

For any $n \geq 0$, denote

$$C(n) = 1 + \sum_{k=1}^{n} \frac{1}{A_{v_0^{(i)}} A_{v_k^{(i)}}} \prod_{j=1}^{k-1} A_{v_j^{(i)}}^{-2} + \frac{C}{A_{v_0^{(i)}}} \prod_{l=1}^{n} A_{v_l^{(i)}}^{-2}. \tag{4.50}$$

Thus $\dfrac{1}{\beta(\rho^{(i)})} \leq C(R_i - 1)$, note also that, since $\xi_2 = \mathbf{E}(A^{-2}) = 1 + \frac{3}{c^2} + \frac{3}{c^4}$, $E(C(n)) \leq c_{34}\xi_2^{n+1}$. Therefore, for any $K \geq 1$,

$$\frac{1}{\sum_{i=1}^{K} \beta(\rho^{(i)})} \leq C(\min_{1\leq i\leq K} R_i - 1).$$

Taking expectation under $\mathbb{P}$ yields (as R_i i.i.d. let R be a r.v. with the common distribution)

$$\begin{aligned}
\mathbb{E}(\frac{1}{\sum_{i=1}^{K} \beta(\rho^{(i)})}) &\leq \mathbb{E}(\mathbb{E}(C(\min_{1\leq i\leq K} R_i - 1)|R_i;\ 1 \leq i \leq K)) \\
&\leq c_{34}\mathbb{E}(\xi_2^{\min_{1\leq i\leq K} R_i}) \leq c_{34} \sum_{n=0}^{\infty} \xi_2^{n+1}\mathbb{P}(R \geq n+1)^K \\
&\leq c_{34} \sum_{n\geq 0} \xi_2^{n+1}\mathbb{E}(\delta_C^{\sum_{k=0}^{n-1}(d(v_k)-2)})^K
\end{aligned}$$

where $\delta_C = \mathbb{P}(\frac{1}{A_\rho \beta_\rho} > C)$. Let $f(s) = \sum_{k\geq 1} q_k s^k$, as $f(s)/s \downarrow q_1$ as $s \downarrow 0$, for any $\varepsilon > 0$, we can take C large enough to ensure $\frac{f(\delta_C)}{\delta_C} \leq q_1(1+\varepsilon)$, thus

$$\mathbb{E}(\frac{1}{\sum_{i=1}^{K} \beta(\rho^{(i)})}) \leq c_{34} \sum_{n\geq 0} \xi_2^{n+1}(\frac{f(\delta_C)}{\delta_C})^{nK} \leq c_{34} \sum_{n\geq 0} \xi_2^{n+1}(q_1(1+\varepsilon))^{nK}.$$

Now take ε such that $q_1(1+\varepsilon) < 1$, then take K large enough such that $\xi_2(q_1(1+\varepsilon))^K < 1$ leads to

$$\mathbb{E}(\frac{1}{\sum_{i=1}^{K} \beta(\rho^{(i)})}) < c_{12} < \infty$$

Similarly, the following also holds

$$\mathbb{E}(\frac{1}{\sum_{i=1}^{K} A_{\rho^{(i)}}\beta(\rho^{(i)})}) < c_{12} < \infty.$$

In particular, if $q_1\zeta_2 < 1$, we can take $K = 1$ and obtained Further, it follows from (4.50) and Chauchy-Schwartz inequality that

$$C(n)^2 \leq (n+2)\left(1 + \sum_{k=1}^{n} \frac{1}{A_{v_0^{(i)}}^2 A_{v_k^{(i)}}^2} \prod_{j=1}^{k-1} A_{v_j^{(i)}}^{-4} + \frac{C}{A_{v_0^{(i)}}} \prod_{l=1}^{n} A_{v_l^{(i)}}^{-4}\right).$$

Thus,

$$\mathbb{E}[C^2(n)] \leq c_{35}(n+2)\xi_4^{n+1}.$$

As soon as $\zeta_4 < \infty$, the previous argument works again to conclude that for K large enough,

$$\mathbb{E}(\frac{1}{\sum_{i=1}^{K} \beta^2(\rho^{(i)})}) + \mathbb{E}(\frac{1}{\sum_{i=1}^{K_0} A_{\rho^{(i)}}^2\beta^2(\rho^{(i)})}) < c_{13} < \infty.$$

□

CHAPTER 5

A RANDOM SCHRÖDINGER OPERATOR ASSOCIATED TO VRJP, ERRW ON FINITE GRAPH

(based on a joint work with C.Sabot and P.Tarrès) [43]

Abstract

We introduce a new exponential family of probability distributions, which can be viewed as a multivariate generalization of the Inverse Gaussian distribution. Considered as the random potential of a Schrödinger operator, this exponential family is related to the random field that gives the mixing measure of the Vertex Reinforced Jump Process, and hence to the mixing measure of the Edge Reinforced Random Walk, the so-called magic formula. In particular, it gives by direct computation the value of the normalizing constants of these mixing measures answering a question raised by Diaconis.

5.1 Introduction

In this paper we introduce a new multivariate exponential family which can be viewed as a multivariate generalization of the inverse Gaussian law. This exponential family is associated to a network of conductances and provides a random field on the vertices of the network, the latter having the remarkable property that the marginals have inverse gaussian law and that the field is decorrelated at distance two.

This exponential family is mainly motivated by the study of two self-interacting processes, namely the Edge Reinforced Random Walk (ERRW) and the closely related Vertex Reinforced Jump Process (VRJP), but we expect that this exponential family could find some applications in different topics, firstly in Bayesian statistics. This exponential family is closely related to the mixing measure of the VRJP and thus to the mixing measure of the ERRW (the so-called 'magic formula'). In particular, this yields an answer to an old question of Diaconis about direct computation of the normalizing constant of the mixing measure of the ERRW.

More precisely, we consider a non-directed finite graph $\mathcal{G} = (V, E)$ with strictly positive conductances $W_{i,j} = W_{j,i}$ on the edges. Denote by Δ^W the discrete Laplace operator associated with the conductance network $(W_{i,j})$ and write $W_i = \sum_{j:\{i,j\}\in E} W_{i,j}$. The exponential family provides a random vector of positive reals $(\beta_j)_{j\in V}$ such that

$$H_\beta := -\Delta^W + (2\beta - W)$$

is a.s. a positive operator (where $2\beta - W$ is the operator of multiplication by $(2\beta_i - W_i)$ and $2\beta - W$ is considered as a random potential). We prove in Theorem 5.3.2 that if the Green function is defined by

$G = (H_\beta)^{-1}$, then the field (e^{u_j}) giving the mixing measure of the VRJP starting from i_0, c.f. [45], is equal in law to $(G(i_0, j)/G(i_0, i_0))$.

This has several consequences. Firstly, it relates the VRJP to a random Schrödinger operator with an explicit random potential with decorrelation at distance 2. Note that Anderson localization was the main motivation of the works of Disertori, Spencer, Zirnbauer ([23, 24]), the supersymmetric fields related to the mixing measure of the VRJP (c.f. [45]) being a toy-model for some supersymmetric fields that appears in the physic literature in connection with random band matrices. Secondly, it gives a way to couple the mixing fields of the VRJP starting from different points. Finally, it yields an answer to an old question of Diaconis about the direct computation of the normalizing constant of the 'magic formula'.

The paper is organized as follows. In Section 5.2, we define the new exponential family of distributions and give its first properties. In section 5.3, we discuss the link between the exponential family and the Vertex reinforced jump processes. In Section 5.4 we consider the ERRW and answer the question of Diaconis. Sections 5.5 and 5.6 provide the proof of the two main results, namely Theorem 5.2.1 and Theorem 5.3.2.

5.2 A new exponential family

We present an exponential family, which seems to be new, and which is a natural multivariate generalization of the Inverse Gaussian family. This exponential family is associated with a network of conductances: let $V = \{1, \dots, N\}$ be a finite set and $(W_{i,j})_{i \neq j}$ be a set of non-negative reals with $W_{i,j} = W_{j,i} \geq 0$. We denote by E the edges associated to the positive $W_{i,j}$, more precisely, we consider the graph $\mathcal{G} = (V, E)$ with $\{i, j\} \in E$ if and only if $W_{i,j} > 0$, and we denote $i \sim j$ if $\{i, j\} \in E$. Let $\mathrm{d}_{\mathcal{G}}$ be the graph distance on $\mathcal{G}$. When A is a symmetric operator on $\mathbb{R}^V$ (also be considered as a $V \times V$ matrix), we write $A > 0$ if A is positive definite, and $|A|$ for its determinant.

Theorem 5.2.1. *Let $P = (P_{i,j})_{1 \leq i,j \leq N}$ be the symmetric matrix given by*

$$P_{i,j} = \begin{cases} 0 & i = j, \\ W_{i,j} & i \neq j. \end{cases}$$

For any $\theta \in \mathbb{R}_+^N$,

$$(\frac{2}{\pi})^{N/2} \int \mathbb{1}_{2\beta - P > 0} e^{-\langle \theta, \beta \rangle} \frac{d\beta}{\sqrt{|2\beta - P|}} = \exp\left(- \sum_{\{i,j\} \in E} W_{i,j} \sqrt{\theta_i \theta_j} \right) \cdot \prod_{i=1}^{N} \frac{1}{\sqrt{\theta_i}} \tag{5.1}$$

where $d\beta = d\beta_1 \cdots d\beta_N$, and $2\beta - P$ is the operator on $\mathbb{R}^V$ defined by

$$[(2\beta - P)f](i) = 2\beta_i f(i) - \sum_{j : j \sim i} W_{i,j} f(j).$$

Definition 5.2.1. *The exponential family of random probability measure $\nu^{W,\theta}(d\beta)$ is defined by*

$$\nu^{W,\theta}(d\beta) = \mathbb{1}_{2\beta - P > 0} (\frac{2}{\pi})^{N/2} \exp\left(-\langle \theta, \beta \rangle + \sum_{\{i,j\} \in E} W_{i,j} \sqrt{\theta_i \theta_j} \right) \frac{\prod_i \sqrt{\theta_i}}{\sqrt{|2\beta - P|}} d\beta.$$

We will simply write ν^W for $\nu^{W,1}$ in the case where $\theta_i = 1$ for all $i \in V$.

The proof of the main formula (5.1) is given in Section 5.5. We deduce from the previous theorem the following simple but important properties of the measure $\nu^{W,\theta}$.

Proposition 5.2.1. *The Laplace transform of $\nu^{W,\theta}$ is*

$$\int e^{-\lambda\beta}\nu^{W,\theta}(d\beta) = \exp\left(-\sum_{\{i,j\}\in E} W_{i,j}(\sqrt{\lambda_i+\theta_i}\sqrt{\lambda_j+\theta_j}-\sqrt{\theta_i\theta_j})\right)\cdot\prod_{i=1}^{n}\sqrt{\frac{\theta_i}{\lambda_i+\theta_i}}$$

Moreover, if β is a random vector with distribution $\nu^{W,\theta}$, then

- *The marginals β_i are such that $\frac{1}{2\beta_i\theta_i}$ is an Inverse Gaussian distribution with parameter $(\frac{1}{\sum_{j\sim i}W_{i,j}\sqrt{\theta_i\theta_j}}, 1)$*
- *If $V_1\subset V$, $V_2\subset V$ are two subsets of V such that $d_{\mathcal{G}}(V_1,V_2)\geq 2$, then $(\beta_i)_{i\in V_1}$ and $(\beta_j)_{j\in V_2}$ are independent.*

The family can be reduced to the case $\theta = 1$ by changing W, as shown in the next corollary.

Corollary 5.2.1. *Let $(\beta_j)_{j\in V}$ be distributed according to $\nu^{W,\theta}$. Then $(\theta\beta)$ is distributed according to ν^{W^θ}, where $W^\theta_{i,j} = W_{i,j}\sqrt{\theta_i\theta_j}$.*

It is clear from the expression of the Laplace transform that if the graph has several connected components then the random field $(\beta_j)_{j\in V}$ splits accordingly into independent random subvectors. Therefore, we will always assume in the sequel that the graph $\mathcal{G}$ is connected.

5.3 The link with Vertex reinforced Jump process

5.3.1 Vertex Reinforced Jump Process, definition and main properties

In this section we explain the link between the exponential family of Section 5.2 and the Vertex reinforced Jump Process (VRJP), which is a linearly reinforced process in continuous time, defined in [18], investigated on trees in [8], and on general graphs by the first two authors in [45]. Consider as in the previous section a conductance network $(W_{i,j})$ and the associated graph $\mathcal{G} = (V, E)$. Fix also some positive parameters $(\phi_i)_{i\in V}$ on the vertices. Assume that the graph $\mathcal{G}$ is connected.

We call VRJP with conductances $(W_{i,j})$ and initial local time (ϕ_i) the continuous-time process $(Y_t)_{t\geq 0}$ on V, starting at time 0 at some vertex $i_0\in V$ and such that, if Y is at a vertex $i\in V$ at time t, then, conditionally on $(Y_s, s\leq t)$, the process jumps to a neighbour j of i at rate

$$W_{i,j}L_j(t),$$

where

$$L_j(t) := \phi_j + \int_0^t \mathbb{1}_{\{Y_s=j\}}\,ds.$$

The following time change, introduced in [45], plays a central role. Let

$$D(t) = \sum_{i\in V}(L_i^2(t)-\phi_i^2), \tag{5.2}$$

define Z_t as the time changed process

$$Z_t = Y_{D^{-1}(t)}.$$

Let $(\ell_j(t))$ be the local time of Z at time t (that is, $\ell_j(t) = \int_0^t \mathbb{1}_{Z_s=j}ds$). Conditionally on the past, at time t, the process Z jumps from $Z_t = i$ to a neighbour j at rate (c.f. [46], Lemma 3)

$$W_{i,j}\sqrt{\frac{\phi_j^2+\ell_j(t)}{\phi_i^2+\ell_i(t)}}.$$

We state below one of the main results of [45], Proposition 1 and Theorem 2. The theorem was stated in [45] in the case $\phi = 1$, this version of the theorem can be deduced by a simple change of time, details are given in Section 5.8.

Theorem 5.3.1. *Assume that $\mathcal{G}$ is finite. Suppose that the VRJP starts at i_0. The limit*

$$U_i = \frac{1}{2}\lim_{t\to\infty}\left(\log(\frac{\ell_i(t)+\phi_i^2}{\ell_{i_0}(t)+\phi_{i_0}^2}) - \log(\frac{\phi_i^2}{\phi_{i_0}^2})\right)$$

exists a.s. and, conditionally on U, Z_t is a mixture of Markov jump processes with jump rates

$$\frac{1}{2}W_{i,j}e^{U_j-U_i}.$$

Moreover (U_j) has the following distribution on $\{(u_i),\ \ u_{i_0}=0\}$

$$Q_{i_0}^{W,\phi}(du) = \frac{\prod_{j\neq i_0}\phi_j}{\sqrt{2\pi}^{|V|-1}}e^{-\sum_{j\in V}u_j}e^{-\frac{1}{2}\sum_{\{i,j\}\in E}W_{i,j}(e^{u_i-u_j}\phi_j^2+e^{u_j-u_i}\phi_i^2-2\phi_i\phi_j)}\sqrt{D(W,u)}\,du, \tag{5.3}$$

with $du = \prod_{j\in V\setminus\{i_0\}}du_j$ and

$$D(W,u) = \sum_T\prod_{\{i,j\}\in T}W_{i,j}e^{u_i+u_j}$$

where the sum runs on the set of spanning trees T of $\mathcal{G}$. We simply write $Q_{i_0}^W$ for $Q_{i_0}^{W,1}$

The fact that the total mass of the measure $Q_{i_0}^{W,\phi}$ is 1 is both a non-trivial and a useful fact, it for example plays a central role in the delocalization and localization results of [23, 24]. In [45], it is a consequence of the fact that it is the probability distribution of the random variables U. In [23], this is proved using a sophisticated supersymmetric argument, the so-called localization principle. In the present paper, we give a direct 'computational' proof of this result, based on the identity (5.1) and on a change of variable that relates the field (u_j) with the random vector (β_j) of Theorem 5.2.1, c.f. forthcoming Theorem 5.3.2.

5.3.2 Relation with the random potential β.

The second main result of this paper gives a way to construct the mixing field e^u defined in the previous subsection from the random potential (β_j) defined in Theorem 5.2.1. It gives also a natural way to couple the mixing measure of VRJP starting from different points.

Theorem 5.3.2. *Let β be a random potential with distribution $\nu^{W,\phi^2}(d\beta)$, c.f. Theorem 5.2.1. If G is the inverse of $(2\beta - P)$, then $(G(i,j))$ has positive coefficients. Define $(u(i,j))_{i,j\in V}$ by*

$$e^{u(i,j)} = \frac{G(i,j)}{G(i,i)}.$$

Then for $i_0\in V$, the function $j\to u(i_0,j)$ is the unique solution $j\mapsto u_j$ of the equation

$$\begin{cases}\sum_{j\sim i}\frac{1}{2}W_{i,j}e^{u_j-u_i} = \beta_i, & i\neq i_0\\ u_{i_0}=0,\end{cases} \tag{5.4}$$

In particular $(u(i_0,j))_{j\in V}$ is $(\beta_j)_{j\in V\setminus\{i_0\}}$ measurable. With this definition we have the following properties

i) *The random field* $(u(i_0, j))_{j\in V}$ *has the distribution of the mixing measure* $Q_{i_0}^{W,\phi}(du)$ *of the VRJP starting from* i_0 *with initial local time* (ϕ).

ii) *The random variable* $G(i_0, i_0)$ *has the distribution of* $1/(2\gamma)$, *where* γ *is a gamma random variable with parameters* $(1/2, 1/\phi_{i_0}^2)$. *Moreover,* $G(i_0, i_0)$ *is independent of* $(\beta_j)_{j\neq i_0}$ *and so also of the field* $(u(i_0, j))_{j\in V}$. *Finally, at the point* i_0 *we have*

$$\beta_{i_0} = \frac{1}{2G(i_0, i_0)} + \sum_{j\,:\,j\sim i_0} \frac{1}{2} W_{i_0,j} e^{u(i_0,j)}.$$

The proof of Theorem 5.3.2 is given in Section 5.6. It is clear from the previous theorem how to construct the random potential β from the field u of Theorem 5.3.1, it is described precisely in the next Corollary.

Corollary 5.3.1. *Consider a VRJP with edge weight* $(W_{i,j})$ *and initial local time* (ϕ), *starting at* i_0. *Let* (u_i) *be distributed according to* $Q_{i_0}^{W,\phi}$ *of Theorem 5.3.1. Let*

$$\tilde{\beta}_i = \frac{1}{2} \sum_{j\,:\,j\sim i} W_{i,j} e^{u_j - u_i},$$

Let γ *be a Gamma distributed random variable with parameters* $(\frac{1}{2}, 1/\phi_{i_0}^2)$, *independent of* (u_j), *and let*

$$\beta_i = \tilde{\beta}_i + \mathbb{1}_{i_0}\gamma.$$

Then β *has the law* ν^{W,ϕ^2} *of Theorem 5.2.1.*

As mentioned in the introduction, Theorem 5.3.2 has several consequences. Firstly it explicitly relates the VRJP to a random Schrödinger operator. Indeed, let $\Delta^W = P - W$ be the discrete Laplacian on the conductance network (where W is the operator of multiplication by $(W_i)_{i\in V}$), then Theorem 5.3.2 relates the mixing measure of the VRJP with the Green function of the random Schrödinger operator $-\Delta^W + v$, where v is the random potential $v_i = 2\beta_i - W_i$. Secondly, it gives a natural coupling between the random fields $(u_j)_{j\in V}$ associated with VRJP starting from different points. Indeed, the exponential family $(\beta_i)_{i\in V}$ gives the same role to each vertex of the graph, and the matrix $(u(i,j))_{i,j\in V}$ couples the mixing measures of the VRJP starting from different points of the graph. Finally, it gives a computational proof of the fact that $\int Q_{i_0}^{W,\theta}(du)x = 1$ for any θ. Indeed, its is a consequence of Theorem 5.2.1, the relation between (β) and (u) being a 'simple' change of variables.

5.4 Edge reinforced random walk

5.4.1 Definition and magic formula

The Edge Reinforced Random Walk (ERRW) is a famous discrete time process introduced in 1986 by Coppersmith and Diaconis, [13]. Let $(a_{i,j})_{\{i,j\}\in E}$ be a set of positive weights on the edges of the graph $\mathcal{G}$, the ERRW is defined as follows.

Let $(X_n)_{n\in\mathbb{N}}$ be a random process that takes values in V, and let $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$ be the filtration of its past. For any $e \in E$, $n \in \mathbb{N}$, let

$$Z_n(e) = a_e + \sum_{k=1}^{n} \mathbb{1}_{\{\{X_{k-1}, X_k\}=e\}} \tag{5.5}$$

be the number of crosses of the (non-directed) edge e up to time n plus the initial weight a_e.

Then $(X_n)_{n\in\mathbb{N}}$ is called Edge Reinforced Random Walk (ERRW) with starting point $i_0 \in V$ and weights $(a_e)_{e\in E}$, if $X_0 = i_0$ and, for all $n \in \mathbb{N}$,

$$\mathbb{P}(X_{n+1} = j \mid \mathcal{F}_n) = \mathbb{1}_{\{j\sim X_n\}} \frac{Z_n(\{X_n, j\})}{\sum_{k\sim X_n} Z_n(\{X_n, k\})}. \tag{5.6}$$

We denote by $\mathbb{P}_{i_0}^{ERRW,(a)}$ the law of the ERRW starting from the initial vertex i_0 and initial weights (a).

A fundamental property of the ERRW, stated in the next theorem, is that on finite graphs the ERRW is a mixture of reversible Markov chains, and the mixing measure can be determined explicitly (the so-called Coppersmith-Diaconis measure, or 'magic formula'). It is a consequence of a de Finetti theorem for Markov chains due to Diaconis and Freedman [20], and the explicit determination of the law is due to Diaconis and Coppersmith, c.f. [16, 31, 35]. It has also applications in Bayesian statistics [7, 6, 21].

Theorem 5.4.1. *[16, 31]*

Assume that $\mathcal{G} = (V, E)$ is a finite graph and set $a_i = \sum_{j:\{i,j\}\in E} a_{i,j}$ for all $i \in V$. Fix an edge e_0 incident to i_0, and define $\mathcal{H}_{e_0} = \{\forall e \in E,\ y_e > 0,\ y_{e_0} = 1\}$ (similarly denote $y_i = \sum_{i\in e} y_e$). Consider the following positive measure defined on $\mathcal{H}_{e_0}$ defined by its density

$$\mathcal{M}_{i_0}^{(a)}(dy) = C(a, i_0)\frac{\sqrt{y_{i_0}}\prod_{e\in E} y_e^{a_e}}{\prod_{i\in V} y_i^{\frac{1}{2}(a_i+1)}}\sqrt{D(y)}\prod_{e\neq e_0}\frac{dy_e}{y_e}, \tag{5.7}$$

where $D(y)$ is any diagonal minor of the matrix $\begin{pmatrix} -y_{v_1} & y_{v_1,v_2} & \cdots & y_{v_1,v_{|V|}} \\ & \cdots & \cdots & \\ y_{v_{|V|},v_1} & \cdots & \cdots & -y_{v_{|V|}} \end{pmatrix}$ *with $V = \{v_1, \dots, v_{|V|}\}$, and where*

$$C(a, i_0) = \frac{2^{1-|V|+\sum_{e\in E} a_e}}{\sqrt{\pi}^{|V|-1}} \cdot \frac{\prod_{i\in V}\Gamma(\frac{1}{2}(a_i + 1 - \mathbb{1}_{i=i_0}))}{\prod_{e\in E}\Gamma(a_e)}$$

Then $\mathcal{M}_{i_0}^{(a)}$ is a probability measure on $\mathcal{H}_{e_0}$, and it is the mixing measure of the ERRW starting from i_0, more precisely

$$\mathbb{P}_{i_0}^{ERRW,(a)}(\cdot) = \int_{\mathcal{H}_{i_0}} P_{i_0}^{(y)}(\cdot) d\mathcal{M}_{i_0}^{(a)}(y).$$

where $P_{i_0}^{(y)}$ denote the reversible Markov chain starting at i_0 with conductance (y).

5.4.2 The question of Diaconis

The fact that $\mathcal{M}_{i_0}^{(a)}(dy)$ is a probability measure is a consequence of the fact that it is the mixing measure of the ERRW. In fact it is obtained as the limit distribution of the normalized function $Z_n(e)$ defined in (5.5) (c.f. [31]):

$$\left(\frac{Z_n(e)}{Z_n(e_0)}\right)_{e\in E} \xrightarrow{law} \mathcal{M}_{i_0}^{(a)}.$$

One question raised by Diaconis is the following

$$\text{(Q) Prove by direct computation that } \int \mathcal{M}_{i_0}^{(a)}(dy) = 1\ ? \tag{5.8}$$

Partial answer to this question was given in the case of the triangle graph by Diaconis and Stong[1].

We provide below an answer to this question. A first simplification comes from [45], where the question was reduced to prove that $\int \mathcal{Q}_{i_0}^W(du) = 1$ where $\mathcal{Q}_{i_0}^W$ is the mixing measure of the VRJP, c.f. Theorem 5.3.1. Indeed, the following was proved in [45], Theorem 1.

Theorem 5.4.2. *Consider (Y_n) the discrete time process associated with the VRJP (Y_t) (i.e. taken at jump times) with conductances $(W_{i,j})$ and $\phi = 1$. Take now the conductances $(W_e)_{e\in E}$ as independent random variables with gamma distribution with parameters $(a_e)_{e\in E}$. Then the 'annealed' law of Y_n (i.e. the law after taking expectation with respect to the random (W_e)) is the law of the ERRW (X_n) with initial weights $(a_e)_{e\in E}$.*

This immediately implies an identity between the mixing measures $\mathcal{M}_{i_0}^{(a)}$ and $\mathcal{Q}_{i_0}^W$: indeed, by Theorem 5.3.1, (Y_n) is a mixture of Markov jump processes with conductances $W_{i,j}e^{u_i+u_j}$, it implies that for all bounded test function ϕ

$$\int_{\mathcal{H}_{e_0}} \phi((y_e))\mathcal{M}_{i_0}^{(a)}(dy) = \int_{\mathbb{R}^E} \prod_{e\in E} \frac{W_e^{a_e-1}e^{-W_e}}{\Gamma(a_e)} \left(\int \phi((W_{i,j}e^{u_i+u_j}))\mathcal{Q}_{i_0}^W(du)\right) dW. \tag{5.9}$$

where $dW = \prod_{e\in E} dW_e$. This identity was checked by direct computation in section 5 of [45]. Now, the fact that $\int \mathcal{Q}_{i_0}^W(du) = 1$ is a consequence of the computation of the integral (5.1) in Theorem 5.2.1 and the change of variable provided by Theorem 5.3.2.

5.5 Proof of inverse Laplace transform

Lemma 5.5.1. *Let $P = (P_{i,j})_{1\le i,j\le n}$ be a symmetric matrix with*

$$P_{i,j} = \begin{cases} 0, & i = j, \\ W_{i,j}, \in \mathbb{R}^+ & i \neq j. \end{cases}$$

Consider any diagonal matrix β with diagonal entry β_i, $i = 1, \dots, n$ such that $M = 2\beta - P$ is positive definite. Define

$$x_i = \frac{M(1,\dots,i|1,\dots,i)}{M(1,\dots,i-1|1,\dots,i-1)},$$

where $M(I|J)$ is the minor of matrix M that corresponds to the rows with index in I and columns with index in J, and define $(H_{i,j})_{i<j}$ recursively by

$$\begin{cases} H_{1,j} = W_{1,j} & j > 1 \\ H_{i,j} = W_{i,j} + \sum_{k=1}^{i-1} \frac{H_{k,i}H_{k,j}}{x_k} & i \ge 2,\ j > i \end{cases}$$

We have

$$x_i = 2\beta_i - \sum_{k=1}^{i-1} \frac{H_{k,i}^2}{x_k},$$

and there exists a lower triangular matrix T with 1 on the diagonal such that

$$TM = \begin{pmatrix} x_1 & -H_{1,2} & \cdots & -H_{1,n} \\ 0 & x_2 & \cdots & -H_{2,n} \\ & \cdots & & -H_{n-1,n} \\ 0 & \cdots & 0 & x_n \end{pmatrix};\ \text{i.e. } [TM]_{i,j} = \begin{cases} x_i & i = j \\ -H_{i,j} & i < j \\ 0 & \text{otherwise} \end{cases}$$

[1] Private communication

Proof. The result follows directly from (2.6) of [52]. In fact, if $M = LU$ is the LU-decomposition of the matrix M, that is, L is a lower triangular matrix having diagonal entries 1 and U is an upper triangular matrix, then

$$L = T^{-1} \text{ and } U = TM.$$

Hence, this lemma gives an explicit description of the matrix U. To be self-contained, we give a proof in Section 5.7. □

Claim 5.5.1. *For any* $\theta, \theta_1, \theta_2 \in \mathbb{R}_+^*$,

$$\int_0^\infty \exp(-\frac{\theta x}{2})\frac{1}{\sqrt{x}}dx = \sqrt{\frac{2\pi}{\theta}},$$
$$\int_0^\infty \exp(-\frac{\theta_1 x}{2} - \frac{\theta_2}{2x})\frac{1}{\sqrt{x}}dx = \exp(-\sqrt{\theta_1\theta_2})\sqrt{\frac{2\pi}{\theta_1}}.$$

Proof. Consequence of the density of Gamma and Inverse Gaussian distributions. □

Proof of Theorem 5.2.1 . By Lemma 5.5.1,

$$\begin{aligned}
\sum_{l=1}^n \theta_l\beta_l &= \sum_{l=1}^n \theta_l(\frac{x_l}{2} + \sum_{k=1}^{l-1}\frac{H_{k,l}^2}{2x_k}) \\
&= \sum_{l=1}^n \frac{\theta_l x_l}{2} + \sum_{l=1}^{n-1}\frac{1}{2x_l}(\sum_{k=l+1}^n \theta_k H_{l,k}^2) \\
&= \frac{\theta_n x_n}{2} + \sum_{l=1}^{n-1}[\frac{\theta_l x_l}{2} + \frac{1}{2x_l}(\sum_{k=l+1}^n \theta_k H_{l,k}^2)].
\end{aligned}$$

Note that

$$\Psi : (x_i, 1 \le i \le n) \mapsto (\beta_i, 1 \le i \le n) \text{ where } \beta_i = \frac{x_i}{2} + \sum_{k=1}^{i-1}\frac{H_{k,i}^2}{x_k}$$

is clearly a bijection from $\mathbb{R}_+^n$ to $\{2\beta - P > 0\}$, moreover, the Jacobian of Ψ is $\frac{1}{2^n}$, hence Ψ is a diffeomorphisms. We first integrate the variable x_n

$$\begin{aligned}
I &:= \int_{M>0}\frac{\exp(-\theta\beta)}{\sqrt{|M|}}d\beta \\
&= \int_{\mathbb{R}_+^n}\exp\left(-\frac{\theta_n x_n}{2} - \sum_{l=1}^{n-1}[\frac{\theta_l x_l}{2} + \frac{1}{2x_l}(\sum_{k=l+1}^n \theta_k H_{l,k}^2)]\right)\frac{1}{\sqrt{x_1\cdots x_n}}\frac{1}{2^n}dx \\
&= \frac{1}{2^n}\int_0^\infty \exp(-\frac{\theta_n x_n}{2})\frac{dx_n}{\sqrt{x_n}}\int_{\mathbb{R}_+^{n-1}}\exp\left(-\sum_{l=1}^{n-1}[\frac{\theta_l x_l}{2} + \frac{1}{2x_l}(\sum_{k=l+1}^n \theta_k H_{l,k}^2)]\right)\frac{dx_{n-1}\cdots dx_1}{\sqrt{x_1\cdots x_{n-1}}} \\
&= \frac{1}{2^n}\sqrt{\frac{2\pi}{\theta_n}}\int_{\mathbb{R}_+^{n-1}}\exp\left(-\sum_{l=1}^{n-1}[\frac{\theta_l x_l}{2} + \frac{1}{2x_l}(\sum_{k=l+1}^n \theta_k H_{l,k}^2)]\right)\frac{dx_{n-1}\cdots dx_1}{\sqrt{x_1\cdots x_{n-1}}}.
\end{aligned}$$

It turns out that the integral over $dx_1 \cdots dx_{n-1}$ can be computed explicitly using Claim 5.5.1, the first step is to note that

$$\begin{aligned}
I_{n-1} &:= \int_{\mathbb{R}_+^{n-1}} \exp\left(- \sum_{l=1}^{n-1} [\frac{\theta_l x_l}{2} + \frac{1}{2x_l}(\sum_{k=l+1}^{n} \theta_k H_{l,k}^2)] \right) \frac{dx_{n-1} \cdots dx_1}{\sqrt{x_1 \cdots x_{n-1}}} \\
&= \int_{\mathbb{R}_+^{n-2}} \int_{\mathbb{R}_+} e^{-\frac{\theta_{n-1} x_{n-1}}{2} - \frac{H_{n-1,n}^2 \theta_n}{2x_{n-1}}} \frac{1}{\sqrt{x_{n-1}}} dx_{n-1} \exp\left(- \sum_{l=1}^{n-2} [\frac{\theta_l x_l}{2} + \frac{1}{2x_l}(\sum_{k=l+1}^{n} \theta_k H_{l,k}^2)] \right) \frac{dx_{n-2} \cdots dx_1}{\sqrt{x_1 \cdots x_{n-2}}} \\
&= \int_{\mathbb{R}_+^{n-2}} \sqrt{\frac{2\pi}{\theta_{n-1}}} e^{-H_{n-1,n}\sqrt{\theta_{n-1}\theta_n}} \exp\left(- \sum_{l=1}^{n-2} [\frac{\theta_l x_l}{2} + \frac{1}{2x_l}(\sum_{k=l+1}^{n} \theta_k H_{l,k}^2)] \right) \frac{dx_{n-2} \cdots dx_1}{\sqrt{x_1 \cdots x_{n-2}}}
\end{aligned}$$

recall that

$$H_{n-1,n} = W_{n-1,n} + \frac{H_{1,n-1}H_{1,n}}{x_1} + \cdots + \frac{H_{n-2,n-1}H_{n-2,n}}{x_{n-2}},$$

hence,

$$\begin{aligned}
I_{n-1} &= \sqrt{\frac{2\pi}{\theta_{n-1}}} e^{-W_{n-1,n}\sqrt{\theta_{n-1}\theta_n}} I_{n-2} \\
I_{n-2} &= \int_{\mathbb{R}_+^{n-2}} \exp(- \sum_{l=1}^{n-2} [\frac{\theta_l x_l}{2} + \frac{1}{2x_l}((\sum_{j=n-1}^{n} H_{l,j}\sqrt{\theta_j})^2 + \sum_{k=l+1}^{n-2} \theta_k H_{l,k}^2)]) \frac{dx_{n-2} \cdots dx_1}{\sqrt{x_1 \cdots x_{n-2}}}.
\end{aligned}$$

In the integral I_{n-2}, we again note that Claim 5.5.1 applies, and we can integrate w.r.t. dx_{n-2} to obtain

$$\begin{aligned}
I_{n-2} &= \sqrt{\frac{2\pi}{\theta_{n-2}}} e^{-\sum_{j=n-1}^{n} W_{n-2,j}\sqrt{\theta_{n-2}\theta_j}} I_{n-3} \\
I_{n-3} &= \int_{\mathbb{R}_+^{n-3}} \exp(- \sum_{l=1}^{n-3} [\frac{\theta_l x_l}{2} + \frac{1}{2x_l}((\sum_{j=n-2}^{n} H_{l,j}\sqrt{\theta_j})^2 + \sum_{k=l+1}^{n-3} \theta_k H_{l,k}^2)]) \frac{dx_{n-3} \cdots dx_1}{\sqrt{x_1 \cdots x_{n-3}}}.
\end{aligned}$$

An ‘immediate’ recurrence shows that, we have for any $1 \le m \le n$

$$I_{n-m} = \int_{\mathbb{R}_+^{n-m}} \exp(- \sum_{l=1}^{n-m} [\frac{\theta_l x_l}{2} + \frac{1}{2x_l}((\sum_{j=n-m+1}^{n} H_{l,j}\sqrt{\theta_j})^2 + \sum_{k=l+1}^{n-m} \theta_k H_{l,k}^2)]) \frac{dx_{n-m} \cdots dx_1}{\sqrt{x_1 \cdots x_{n-m}}}.$$

In particular

$$I_1 = \sqrt{\frac{2\pi}{\theta_1}} \exp(- \sum_{j=2}^{n} W_{1,j}\sqrt{\theta_1\theta_j}).$$

Therefore,

$$
\begin{aligned}
I &= \frac{1}{2^n}\sqrt{\frac{2\pi}{\theta_n}} I_{n-1} \\
&= \frac{1}{2^n}\sqrt{\frac{2\pi}{\theta_n}}\sqrt{\frac{2\pi}{\theta_{n-1}}} \exp(-W_{n-1,n}\sqrt{\theta_{n-1}\theta_n}) I_{n-2} \\
&= \cdots \\
&= \frac{1}{2^n}\sqrt{\frac{(2\pi)^k}{\theta_n\cdots\theta_{n-k+1}}} \exp(-\sum_{i>j>n-k} W_{i,j}\sqrt{\theta_i\theta_j}) I_{n-k} \\
&= \cdots \\
&= \frac{1}{2^n}\sqrt{\frac{(2\pi)^n}{\theta_n\cdots\theta_1}} \exp(-\sum_{i\sim j} W_{i,j}\sqrt{\theta_i\theta_j}).
\end{aligned}
$$

□

Proof of Proposition 5.2.1. By Theorem 5.2.1

$$
\begin{aligned}
\int e^{-\lambda\beta} d\nu^{W,\theta} &= \int_{2\beta-P>0} (\frac{2}{\pi})^{N/2} e^{-(\theta+\lambda)\beta} \frac{d\beta}{\sqrt{|2\beta-P|}} e^{\sum_{i\sim j} W_{i,j}\sqrt{\theta_i\theta_j}} \prod_i \sqrt{\theta_i} \\
&= \exp(-\sum_{\{i,j\}\in E} W_{i,j}(\sqrt{(\theta_i+\lambda_i)(\theta_j+\lambda_j)} - \sqrt{\theta_i\theta_j}))\cdot \prod_i \frac{\sqrt{\theta_i}}{\sqrt{\theta_i+\lambda_i}}
\end{aligned}
$$

Therefore, the β_i marginal has Laplace transform

$$
\mathbb{E}(\exp(-\lambda\beta_i)) = \exp(-\sum_{j:j\sim i} W_{i,j}\sqrt{\theta_i\theta_j}(\sqrt{1+\lambda/\theta_i}-1))\frac{1}{\sqrt{1+\lambda/\theta_i}}
$$

Note that if $X \sim IG(\frac{1}{\sum_{j\sim i} W_{i,j}\sqrt{\theta_i\theta_j}}, 1)$ then

$$
\mathbb{E}(\exp(-\frac{\lambda}{2X\theta_i})) = \exp(-\sum_{j:j\sim i} W_{i,j}\sqrt{\theta_i\theta_j}(\sqrt{1+\lambda/\theta_i}-1))\frac{1}{\sqrt{1+\lambda/\theta_i}}
$$

Finally the independence stem from the expression of Laplace transform. □

5.6 Proof of results relating the VRJP and β

Proof of Theorem 5.3.2. Fix $i_0 \in V$. Let us first justify the existence and uniqueness of $u(i_0, i)$ defined by the linear system (5.4). As $(2\beta - P)$ is an M-matrix, its inverse G satisfies $G(i,j) > 0$ for any i, j. A solution (u_j) of equation (5.4) is necessarily of the form $e^{u_j} = 2\gamma G(i_0, j)$ for some constant $\gamma \in \mathbb{R}$. The normalization $u_{i_0} = 0$ implies $\gamma = \frac{1}{2G(i_0,i_0)}$. Hence the unique solution of the system (5.4) is given by $u_j = u(i_0, j)$ defined in Theorem 5.3.2.

Denote

$$
\mathcal{D} = \{(\beta_i)_{i\in V} \in (\mathbb{R}_+ \setminus \{0\})^V,\ \ 2\beta - P > 0\}.
$$

We consider the transformation

$$\begin{array}{rccl}\Phi: & \mathcal{D} & \to & \{(u_j)_{j\in V}\in\mathbb{R}^V,\ u_{i_0}=0\}\times\mathbb{R}_+^* \\ & (\beta) & \mapsto & ((u_j),\gamma),\end{array}$$

where (u_j) is the unique solution of the system (5.4) and $\gamma=\frac{1}{2G(i_0,i_0)}$. We first prove that Φ is a diffeomorphism. By the previous argument it is well-defined and injective. Reciprocally, starting from $((u_j),\gamma)$ on the right hand side, we define (β_i) by

$$\beta_i=\sum_{j\sim i}\frac{1}{2}W_{i,j}e^{u_j-u_i}+\mathbb{1}_{i=i_0}\gamma.$$

It is clear that with this definition, (u_j) is the solution of (5.4) with (β_j). It remains to prove that $2\beta-P>0$: it is a consequence Theorem (2.3)- (J30) of [10]:

Proposition 5.6.1. *Let $A\in Z_n=\{M\in M_n(\mathbb{R}),\ m_{i,j}\le 0,\ \text{if } i\neq j\}$, A is positive stable*[2] *if and only if there eixsts $\xi\gg 0$*[3] *with $A\xi>0$*[4] *and*

$$\sum_{j=1}^{k}a_{k,j}\xi_j>0,\ k=1,\dots,n. \tag{5.10}$$

Apply this Proposition with $A=2\beta-P$ and $\xi=e^{u_\cdot}$. The conditions $\xi\gg 0$ and $A\xi>0$ are easily verified. For (5.10), we need to arrange the vertices in such a way that, $n=i_0$ and that for any vertex k, there are some $l>k$ such that $W_{k,l}>0$. This can always be done since the graph is connected. Indeed, take any spanning tree of the graph, consider the distance of each vertex to i_0. Label the vertices by $1,2,\dots$ starting from the most distance vertices, if several vertices have the same distances to i_0, label them in any arbitrary way. It can be verify that such labelling is decreasing for the spanning tree rooted at i_0. Hence we obtain (5.10) and $2\beta-P>0$.

We now make the change of variable given by Φ^{-1}, and we will prove that if (β) follows the law ν^{W,ϕ^2}, then $(u,\gamma)=\Phi^{-1}(\beta)$ follows the law $\mathcal{Q}_{i_0}^{W,\phi}\otimes\Gamma(\frac{1}{2},\frac{1}{\phi_{i_0}^2})$.

Let J be the Jacobian matrix of Φ (i.e. $J_{i,j}=\frac{\partial\beta_i}{\partial u_j}, j\neq i_0$), then

$$J_{i,j}=\begin{cases}\delta_{i,i_0} & \text{if } j=i_0,\\ \frac{1}{2}W_{i,j}e^{u_j-u_i} & \text{if } i\neq j,\ j\neq i_0,\\ -\beta_i & \text{if } i=j\neq i_0.\end{cases}$$

We can factorize the ith row of J by e^{-2u_i} for each i, then develope the resulting matrix according to the i_0th column, and we found that

$$|J|=\frac{1}{2^{|V|-1}}e^{-2\sum_i u_i}D(W,u)$$

On the other hand,

$$|2\beta-P|=2\gamma e^{-2\sum_i u_i}D(W,u).$$

[2] All its eigenvalue have positive real part.

[3] $\xi\gg\eta$ means for any coordinate i, $\xi_i>\eta_i$

[4] $\xi>0$ means $\xi_i\ge 0$ and $\xi\neq 0$

Let ψ be a positive test function. We have

$$\begin{aligned}
&\int \psi(u,\gamma)\nu^{W,\phi^2}(d\beta)\\
&= \int \psi(u,\gamma)2^{|V|/2}\frac{\prod_i \phi_i}{\pi^{|V|/2}}\frac{\exp(-\sum_i \beta_i\phi_i^2+\sum_{\{i,j\}\in E}W_{i,j}\phi_i\phi_j)}{\sqrt{2\gamma e^{-2\sum_i u_i}D(W,u)}}\frac{1}{2^{|V|-1}}e^{-2\sum_i u_i}D(W,u)dud\gamma\\
&= \int \psi(u,\gamma)\frac{\prod_i \phi_i}{(2\pi)^{(|V|-1)/2}}e^{-\sum_i u(i_0,i)}e^{-\frac{1}{2}\sum_{i\sim j}W_{i,j}(e^{u_i-u_j}\phi_j^2+e^{u_j-u_i}\phi_i^2-2\phi_i\phi_j)}\sqrt{D(W,u)}\cdot\frac{e^{-\phi_{i_0}^2\gamma}}{\sqrt{\pi\gamma}}dud\gamma\\
&= \int \psi(u,\gamma)Q_{i_0}^{W,\phi}(du)\frac{\phi_{i_0}e^{-\phi_{i_0}^2\gamma}}{\sqrt{\pi\gamma}}d\gamma.
\end{aligned}$$

This concludes the proof of Theorem 5.3.2 and of Corollary 5.3.1. □

'Computational' proof of Theorem 5.4.1. First we give a direct proof that $\int dQ_{i_0}^{W,\phi}(u) = 1$. By Theorem 5.3.2, and Theorem 5.2.1

$$\begin{aligned}
\int dQ_{i_0}^{W,\phi}(u) &= \int \frac{\phi_{i_0}}{\sqrt{\pi\gamma_{i_0}}}e^{-\gamma_{i_0}\phi_{i_0}^2}d\gamma_{i_0}\int dQ_{i_0}^{W,\phi}(u)\\
&= \int \mathbb{1}_{2\beta-P>0}\frac{2^{|V|}}{(2\pi)^{|V|/2}}\frac{\exp(-\sum_i \beta_i+\sum_{\{i,j\}\in E}W_{i,j}\phi_i\phi_j)}{\sqrt{|2\beta-P|}}d\beta = 1
\end{aligned}$$

In Section 5 of [45] we proved by direct computation that (cf Equation (5.9))

$$\int_{\mathcal{H}_{e_0}}\phi((y_e))\mathcal{M}_{i_0}^{(a)}(dy) = \int_{(\mathbb{R}_+)^E}\prod_{e\in E}\frac{W_e^{a_e-1}e^{-W_e}}{\Gamma(a_e)}\left(\int \phi((W_{i,j}e^{u_i+u_j}))Q_{i_0}^{W,1}(du)\right)dW. \tag{5.11}$$

It implies that

$$\int_{y_{e_0}=1} d\mathcal{M}_{i_0}^a(y) = 1.$$

This fact can be used to prove directly that $\mathcal{M}_{i_0}^a(dy)$ is the mixing measure of the ERRW starting from initial condition (a) and initial vertex i_0. Indeed, for any finite path $\sigma : i_0 \to i_1 \to \cdots \to i_n$, denote

$$N(i) = |\{k;\ 0\le k\le n-1,\ X_k = i\}|,$$

the number of times vertex i is crossed by the path σ before time $n-1$, similarly

$$N(e) = |\{k;\ 0\le k\le n-1,\ \{X_k, X_{k+1}\} = e\}|,$$

is the number of times the (non-directed) edge e is crossed. The probability of σ for the reversible Markov chain of conductance y is

$$p_{i_0}^y(\sigma) = \frac{\prod_{e\in E} y_e^{N(e)}}{\prod_{i\in V} y_i^{N(i)}}$$

The Integration of $p_{i_0}^y(\sigma)$ w.r.t. $d\mathcal{M}_{i_0}^a(y)$ can be computed by changing the constant $\Gamma(a_e)$ to $\Gamma(a_e+N_e)$ and $\Gamma(\frac{1}{2}(a_i+1))$ to $\Gamma(\frac{1}{2}(a_i+1)+N_i)$. Using the property $\Gamma(x+1) = x\Gamma(x)$ and the notation $(a,n) = \prod_{k=0}^{n-1}(a+k)$, we have

$$\int p_{i_0}^y(\sigma)d\mathcal{M}_{i_0}^a(y) = \frac{\prod_e (a_e, N(e))}{\prod_i (a_i, N(i))}$$

which is the probability of an ERRW to follow the path σ. □

5.7 Proof of the technical lemma

Proof. We will perform successive Gauss elimination on M to make it upper triangular. Denote $l_1, \dots, l_n$ the rows of current matrix. First of all we can write

$$M = M^{(1)} = \begin{pmatrix} x_1^{(1)} & -H_{1,2}^{(1)} & \cdots & -H_{1,n}^{(1)} \\ -H_{1,2}^{(1)} & x_2^{(1)} & \cdots & -H_{2,n}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ -H_{1,n}^{(1)} & -H_{n,2}^{(1)} & \cdots & x_n^{(1)} \end{pmatrix}$$

where we simply denote for any i, $x_i^{(1)} = 2\beta_i$ and for any i, j, $H_{i,j}^{(1)} = W_{i,j}$. Perform $l_2 \leftarrow l_2 + \frac{H_{1,2}^{(1)}}{x_1^{(1)}} l_1, \dots, l_n \leftarrow l_n + \frac{H_{1,n}^{(1)}}{x_1^{(1)}} l_1$ to $M^{(1)}$, we obtain

$$T_1 M = M^{(2)} = \begin{pmatrix} x_1^{(1)} & -H_{1,2}^{(1)} & -H_{1,3}^{(1)} & \cdots & -H_{1,n}^{(1)} \\ 0 & x_2^{(2)} & -H_{2,3}^{(2)} & \cdots & -H_{2,n}^{(2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -H_{2,n}^{(2)} & \cdots & \cdots & x_n^{(2)} \end{pmatrix} \text{ where } [T_1]_{i,j} = \begin{cases} 1 & i = j \\ \frac{H_{1,i}^{(1)}}{x_1^{(1)}} & i > j = 1 \\ 0 & \text{otherwise} \end{cases}$$

and for any $i \geq 2$, $x_i^{(2)} = x_i^{(1)} - \frac{(H_{1,i}^{(1)})^2}{x_1^{(1)}}$ and for any $i, j \geq 2$, $H_{i,j}^{(2)} = H_{i,j}^{(1)} + \frac{H_{1,i}^{(1)} H_{1,j}^{(1)}}{x_1^{(1)}}$.

Suppose by recurrence that at step k we have,

$$M^{(k)} = \begin{pmatrix} x_1^{(1)} & -H_{1,2}^{(1)} & \cdots & \cdots & \cdots & \cdots & \cdots & -H_{1,n}^{(1)} \\ 0 & x_2^{(2)} & -H_{2,3}^{(2)} & & & & & -H_{2,n}^{(2)} \\ \vdots & 0 & \ddots & \ddots & & & & \vdots \\ \vdots & & \ddots & x_{k-1}^{(k-1)} & -H_{k-1,k}^{(k-1)} & \cdots & \cdots & -H_{k-1,n}^{(k-1)} \\ \vdots & & & 0 & x_k^{(k)} & -H_{k,k+1}^{(k)} & \cdots & -H_{k,n}^{(k)} \\ \vdots & & & \vdots & -H_{k,k+1}^{(k)} & \ddots & \ddots & \vdots \\ \vdots & & & \vdots & \vdots & & \ddots & -H_{n-1,n}^{(k)} \\ 0 & 0 & \cdots & 0 & -H_{k,n}^{(k)} & \cdots & -H_{n-1,n}^{(k)} & -x_n^{(k)} \end{pmatrix}$$

perform $l_{k+1} \leftarrow l_{k+1} + \frac{H_{k,k+1}^{(k)}}{x_k^{(k)}} l_k, \dots, l_n \leftarrow l_n + \frac{H_{k,n}^{(k)}}{x_k^{(k)}} l_k$ to $M^{(k)}$, we obtain

$$T_k M^{(k)} = M^{(k+1)} = \begin{pmatrix} x_1^{(1)} & -H_{1,2}^{(1)} & \cdots & \cdots & \cdots & \cdots & \cdots & -H_{1,n}^{(1)} \\ 0 & x_2^{(2)} & -H_{2,3}^{(2)} & & & & & -H_{2,n}^{(2)} \\ \vdots & 0 & \ddots & \ddots & & & & \vdots \\ \vdots & & \ddots & x_k^{(k)} & -H_{k,k+1}^{(k)} & \cdots & \cdots & -H_{k,n}^{(k)} \\ \vdots & & & 0 & x_{k+1}^{(k+1)} & -H_{k+1,k+2}^{(k+1)} & \cdots & -H_{k+1,n}^{(k+1)} \\ \vdots & & & \vdots & -H_{k+1,k+2}^{(k+1)} & \ddots & \ddots & \vdots \\ \vdots & & & \vdots & \vdots & & \ddots & -H_{n-1,n}^{(k+1)} \\ 0 & 0 & \cdots & 0 & -H_{k+1,n}^{(k+1)} & \cdots & -H_{n-1,n}^{(k+1)} & -x_n^{(k+1)} \end{pmatrix}$$

where

$$[T_k]_{i,j} = \begin{cases} 1 & i = j \\ \frac{H^{(k)}_{k,i}}{x^{(k)}_k} & i > j = k \\ 0 & \text{otherwise} \end{cases}$$

and where for any $i \geq k+1$, $x^{(k+1)}_i = x^{(k)}_i - \frac{(H^{(k)}_{k,i})^2}{x^{(k)}_k}$ and for any $i, j \geq k+1$, $H^{(k+1)}_{i,j} = H^{(k)}_{i,j} + \frac{H^{(k)}_{k,i}H^{(k)}_{k,j}}{x^{(k)}_k}$. The result follows from the step n, where we have

$$T_{n-1}M^{(n-1)} = M^{(n)} = \begin{pmatrix} x^{(1)}_1 & -H^{(1)}_{1,2} & \cdots & -H^{(1)}_{1,n} \\ 0 & x^{(2)}_2 & \cdots & -H^{(2)}_{2,n} \\ & \cdots & & -H^{(n-1)}_{n-1,n} \\ 0 & \cdots & 0 & x^{(n)}_n \end{pmatrix}$$

Hence, it gives the LU-decomposition of M where $L^{-1} = T = T_{n-1}T_{n-2}\cdots T_1$ and $U = M^{(n)}$. To recover the lemma, it is enought to identify

$$\begin{cases} x^{(i)}_i = x_i & i = 1, \ldots, n \\ H^{(i)}_{i,j} = H_{i,j} & i < j \end{cases}$$

Finally on the diagonal we see that $x_i = 2\beta_i - \sum_{k=1}^{i-1} \frac{H^2_{k,i}}{x_k}$. □

5.8 Change of time

Let Y_s be the VRJP with edge weight (W) and initial local time $(\phi_i)_{i\in V}$ defined in Section 5.3. Recall that $L_i(t) = \phi_i + \int_0^t \mathbb{1}_{Y_s=i}ds$. Consider the increasing functional $A(s) = \sum_i(\frac{L_i(s)}{\phi_i} - 1)$, and the time-changed process $\tilde{Y}_{\tilde{s}} = Y_{A^{-1}(\tilde{s})}$. Let

$$\tilde{L}_i(\tilde{s}) = 1 + \int_0^t \mathbb{1}_{\tilde{Y}_{\tilde{s}}=i} d\tilde{s}.$$

We always denote by $\tilde{s}$ the time scale of $\tilde{Y}$, we can write in a short way

$$\tilde{s} = A(s), \quad d\tilde{s} = \frac{ds}{\phi_{Y_s}}, \quad L_i(\tilde{s}) = \frac{1}{\phi_i}L_i(s).$$

Obviously, $\tilde{Y}$ is a VRJP with edge weight $W_{i,j}\phi_i\phi_j$ and initial local local time 1 : that is, conditionally on $\mathcal{F}^{\tilde{Y}}_{\tilde{s}}$, $\tilde{Y}$ jumps from i to j at rate

$$W_{i,j}\phi_i\phi_j\tilde{L}_j(\tilde{s}).$$

Note for simplicity

$$W^{\phi}_{i,j} = W_{i,j}\phi_i\phi_j.$$

We can apply [45] Theorem 2. Let

$$\tilde{D}(\tilde{s}) = \sum_i \tilde{L}_i(\tilde{s})^2 - 1,$$

and set $\tilde{Z}_{\tilde{t}} = \tilde{Y}_{\tilde{D}^{-1}(\tilde{t})}$, with local time $\tilde{\ell}_i(\tilde{t}) = \int_0^{\tilde{t}} \mathbb{1}_{\tilde{X}_u = i} du$. By proposition 1 of [45] translated in time scale L (cf relation (2.1) of [45]), we have that $\log \tilde{L}_i(\tilde{s}) - \frac{1}{N} \sum_{j\in V} \log \tilde{L}_j(\tilde{s})$ converges a.s. when $\tilde{s} \to \infty$ to a random vector with distribution given by (3.1) of theorem 1 of [45], where the weights $(W_{i,j})$ are replaced by $(W^{\phi}_{i,j})$. Changing to variables $u_i \to u_i - u_{i_0}$, we get that

$$\lim_{\tilde{s}\to\infty} \log \tilde{L}_i(\tilde{s}) - \log \tilde{L}_{i_0}(\tilde{s}) = U_i$$

exists and has distribution

$$Q^{W^\phi}_{i_0}(du) = \frac{1}{\sqrt{2\pi}^{N-1}} e^{-\sum_{j\in V} u_j} e^{-\frac{1}{2}\sum_{i\sim j} W^{\phi}_{i,j}(\cosh(u_i-u_j)-1)} \sqrt{D(W^\phi, u)}\, du,$$

and that $\tilde{Z}$ is a mixture of Markov Jump Process with jumping rates $\frac{1}{2} W^{\phi}_{i,j} e^{U_j - U_i}$. We need to come back to (Z_t). Recall that $Z_t = Y_{D^{-1}(t)}$, where $D(t)$ is defined in (5.2). From this we have

$$\tilde{t} = \tilde{D}(A(D^{-1}(t))),$$

and

$$d\tilde{t} = \frac{1}{\phi_{\tilde{Y}_{\tilde{s}}}} \frac{\tilde{L}_{\tilde{Y}_{\tilde{s}}}(\tilde{s})}{L_{Y_s}(s)} dt = \frac{1}{\phi^2_{Z_t}} dt.$$

This implies that (Z_t) is a mixture of Markov Jump processes with jumping rates $\frac{1}{2} W_{i,j} e^{U_j + \log\phi_j - U_i - \log\phi_i}$. By simple change of variables, $U_i + \log\phi_i - \log\phi_{i_0}$ has distribution

$$Q^{W,\phi}_{i_0}(du) = \frac{\prod_{j\neq i_0}\phi_j}{\sqrt{2\pi}^{N-1}} e^{-\sum_{j\in V} u_j} e^{-\frac{1}{2}\sum_{i\sim j} W_{i,j}(e^{u_i-u_j}\phi_j^2 + e^{u_j-u_i}\phi_i^2 - 2\phi_i\phi_j)} \sqrt{D(W,u)}\, du.$$

CHAPTER 6

A REPRESENTATION OF ERRW AND VRJP ON INFINITE GRAPH BY RANDOM SCHRÖDINGER OPERATOR

(based on a joint work with C.Sabot) [47]

Abstract

This paper concerns the Vertex reinforced jump process (VRJP) and the Edge reinforced random walk (ERRW) and their link with a random Schrödinger operator. On infinite graphs, we define a 1-dependent random potential β extending that defined in [45] on finite graphs, and consider its associated random Schrödinger operator H_β. We construct a random function ψ as a limit of martingales, such that $\psi = 0$ when the VRJP is recurrent, and ψ is a positive generalized eigenfunction of the random Schrödinger operator with eigenvalue 0, when the VRJP is transient. Then we prove a representation of the VRJP as a mixture of Markov jump processes involving the function ψ, the Green function of the random Schrödinger operator and an independent Gamma random variable. On $\mathbb{Z}^d$, we deduce from this representation a zero-one law for recurrence or transience of VRJP and ERRW, and a functional central limit theorem for VRJP and ERRW at weak reinforcement in dimension $d \geq 3$, using estimates of [22, 23]. We also deduce recurrence of the ERRW in dimension $d = 2$ for any initial constant weights, using the estimates of Merkl and Rolles, [37]. We conjecture some links between recurrence/transience of the VRJP and localization/delocalization of the random Schrödinger operator H_β.

6.1 Introduction

This paper concerns the Vertex Reinforced Jump Process (VRJP) and its relation with a random Schrödinger operator associated with a stationary 1-dependent random potential (i.e. the potential is independent at distance larger or equal to 2).

The VRJP is a continuous time self-interacting process introduced in [19], investigated on trees in [15, 8] and on general graphs in [45, 46]. We first recall its definition. Let $G = (V, E)$ be a non-directed graph with finite degree at each vertex. We write $i \sim j$ if $i \in V$, $j \in V$ and $\{i, j\}$ is an edge of the graph. We always assume that the graph is connected and has no trivial loops (i.e. vertex i such that $i \sim i$). Let $(W_{i,j})_{i\sim j}$ be a set of positive conductances, $W_{i,j} > 0$, $W_{i,j} = W_{j,i}$. The VRJP is the continuous-time process $(Y_s)_{s\geq 0}$ on V, starting at time 0 at some vertex $i_0 \in V$, which, conditionally on the past at time s, if $Y_s = i$, jumps to a neighbour j of i at rate

$$W_{i,j} L_j(s),$$

where

$$L_j(s) := 1 + \int_0^s \mathbb{1}_{\{Y_u=j\}}\, du.$$

In [45], Sabot and Tarrès introduced the following time change of the VRJP

$$Z_t = Y_{D^{-1}(t)},$$

where $D(s)$ is the following increasing function

$$D(s) = \sum_{i\in V}(L_i^2(s) - 1). \tag{6.1.1}$$

As it appears in [45], this is in fact the good time-scale of the VRJP. We denote $\mathbb{P}_{i_0}^{\mathrm{VRJP}}$ the law of (Z_t) starting from the vertex i_0. When the graph is finite it is proved in [45] theorem 2, that the time-changed VRJP Z is a mixture of Markov jump processes. More precisely, there exists a random field $(u_j)_{j\in V}$ such that Z is a mixture of Markov jump processes with jump rates from i to j

$$\frac{1}{2}W_{i,j}e^{u_j-u_i}.$$

The law of the field (u_j) is explicit, cf [45] Theorem 2 and forthcoming Theorem B. It appears to be a marginal of a supersymmetric sigma-field which had been investigated previously by Disertori, Spencer, Zirnbauer (cf [24], [23], [56]). As a consequence of this representation and of [24], [23], it was proved in [45] the following : when the graph has bounded degree, there exists a $0 < \lambda_0$ such that if $W_{i,j} \le \lambda_0$ then the VRJP is positively recurrent, more precisely, Z is a mixture of positive recurrent Markov Jump processes. When the graph is the grid $\mathbb{Z}^d$, with $d \ge 3$, there exists $\lambda_1 < +\infty$ such that if $W_{i,j} \ge \lambda_1$, the VRJP is transient. Hence, it shows a phase transition between recurrence and transience in dimension $d \ge 3$. The question of the representation of the VRJP on infinite graph as a mixture of Markov jump processes is non trivial, especially in the transient case. It is possible to prove such a representation by a weak convergence argument, following [36], but it gives only few information on the mixing law. In this paper we prove such a representation involving the Green function and a generalized eigenfunction of a random Schrödinger operator.

Let us give a flavor of the main results of the paper in the case of the VRJP on $\mathbb{Z}^d$ with $W_{i,j} = W$ constant. We construct a positive 1-dependent random potential $(\beta_j)_{j\in\mathbb{Z}^d}$ (i.e. two subset of the β's are independent if their indices are at least at distance 2) and with marginal given by inverse of Inverse Gaussian law with parameters $1/(dW)$. This field is a natural extension to infinite graphs of the field defined by Sabot, Tarrès, Zeng in [43]. We consider the random Schrödinger operator

$$H_\beta = -W\Delta + V,$$

where Δ is the usual discrete (non-positive) Laplacian and V is the multiplication operator by $V_j = 2\beta_j - 2dW$. Hence, it corresponds to the Anderson model with a random potential which is not i.i.d. but only stationary and 1-dependent. When the VRJP is transient we prove that there exists a positive generalized eigenfunction ψ of H_β with eigenvalue 0, stationary and ergodic. Let $(G(i,j))_{i\in Z^d, j\in\mathbb{Z}^d}$ be defined by

$$G(i,j) = \hat{G}(i,j) + \frac{1}{2}\gamma^{-1}\psi(i)\psi(j),$$

where $\hat{G} = (H_\beta)^{-1}$ is the Green function (which happens to be well-defined) and γ is an extra random variable independent of the field β with law $Gamma(\frac{1}{2})$. We prove the following representation for the

VRJP : the time-changed VRJP Z starting from the point i_0 is a mixture of Markov jump processes with jump rates from i to j

$$\frac{1}{2}W_{i,j}\frac{G(i_0,j)}{G(i_0,i)}$$

When the VRJP is recurrent the same representation is valid with $\psi = 0$. In fact, the function ψ is the a.s. limit of a martingale, the limit being positive when the VRJP is transient and 0 when the VRJP is recurrent. It is remarkable that when the VRJP is recurrent it can be represent as a mixture with β-measurable jump rates, but when the VRJP is transient it involves an extra independent Gamma random variable. This representation extends to infinite graphs the representation given in [43] for finite graphs. The interesting new feature appears in the transient case, where the generalized eigenfunction ψ gets involved in the representation. We suspect that recurrence/transience of the VRJP is related to localization/delocalization of the random Schrödinger operator H_β at the bottom of the spectrum.

We deduce from that representation a functional central limit theorem for the discrete time process associated with the VRJP and with the Edge Reinforced Random Walk in dimension $d \geq 3$ at weak disorder (i.e. for large W for the VRJP and for large initial weights for the ERRW), using the estimates of [23, 22]. Using the polynomial estimate provided by Merkl and Rolles, [37], we are able to prove recurrence of ERRW on $\mathbb{Z}^2$ for all initial constant weights.

6.2 Statements of the results

6.2.1 Representation of the VRJP on infinite graphs

Let $\mathcal{G} = (V, E)$ be a non oriented, locally finite, connected graph without trivial loop. For $i, j \in V$, write $i \sim j$ if i is a neighbor of j. For each edge $e = \{i, j\} \in E$, we associate $W_{i,j} > 0$, some positive real number as the conductance of e. We write $\mathrm{d}_\mathcal{G}$ for the graph distance in $\mathcal{G}$, and for two subsets U, U' of V, define $\mathrm{d}_\mathcal{G}(U, U') = \inf_{i\in U, j\in U'} \mathrm{d}_\mathcal{G}(i, j)$.
Convention : We adopt the notation $\sum_{i\sim j}$ for the sum on all non-oriented edges $\{i, j\}$, counting only once each edge.

Proposition 6.2.1. *There exists a family of positive random variables $(\beta_i)_{i\in V}$, such that for any finite subset $U \subset V$, and $(\lambda_i)_{i\in U} \in \mathbb{R}_+^U$*

$$\mathbb{E}\left(e^{-\sum_{i\in U}\lambda_i\beta_i}\right) = e^{-\sum_{i\sim j,\, i,j\in U} W_{i,j}(\sqrt{(1+\lambda_i)(1+\lambda_j)}-1)-\sum_{i\sim j, i\in U, j\notin U} W_{i,j}(\sqrt{1+\lambda_i}-1)}\frac{1}{\prod_{i\in U}\sqrt{1+\lambda_i}}.$$

In particular, $(\beta_i)_{i\in V}$ has the following properties

- *It is 1-dependent : if $U, U' \subset V$ are such that $\mathrm{d}_\mathcal{G}(U, U') \geq 2$, then $(\beta_i)_{i\in U}$ and $(\beta_j)_{j\in U'}$ are independent.*
- *The marginal β_i is such that $\frac{1}{2\beta_i}$ is an Inverse Gaussian with parameter $(\frac{1}{W_i}, 1)$ where $W_i = \sum_{j\sim i} W_{i,j}$.*

We denote by $\nu^{\mathcal{G},W}(d\beta)$ its distribution.

Remark 6.2.1. *This random field extends to infinite graphs the random field defined in [43]. On finite graphs, its law is explicit, cf [43], Theorem 1, and Theorem C below.*

We call a *path* in $\mathcal{G}$ from i to j a finite sequence $\sigma = (\sigma_0, \dots, \sigma_m)$ in V such that $\sigma_0 = i$, $\sigma_m = j$ and $\sigma_k \sim \sigma_{k+1}$, for $k = 0, \dots, m-1$. The length of σ is defined by $|\sigma| = m$. For such a path we define

$$W_\sigma = \prod_{k=0}^{m-1} W_{\sigma_k,\sigma_{k+1}}, \quad (2\beta)_\sigma = \prod_{k=0}^{m}(2\beta_{\sigma_k}), \quad (2\beta)^-_\sigma = \prod_{k=0}^{m-1}(2\beta_{\sigma_k}). \tag{6.2.2}$$

For the trivial path $\sigma = (\sigma_0)$, we define $W_\sigma = 1$, $(2\beta)_\sigma = 2\beta_{\sigma_0}$, $(2\beta_\sigma)^- = 1$.

Let V_n be an increasing sequence of finite connected subsets of V such that

$$\cup_{n=0}^{\infty} V_n = V.$$

For $i, j \in V_n$, we denote by $\mathcal{P}_{i,j}^{(n)}$ the set of paths σ in V_n going from i to j. Similarly, we denote by $\bar{\mathcal{P}}_i^{(n)}$, the set of paths $\sigma = (\sigma_0, \dots, \sigma_m)$ from $i \in V_n$ to a point $j \notin V_n$ and $\sigma_0, \dots, \sigma_{m-1}$ in V_n.

Definition 6.2.1. *We define for i, j in V*

$$\hat{G}^{(n)}(i,j) = \begin{cases} \sum_{\sigma\in\mathcal{P}_{i,j}^{(n)}} \frac{W_\sigma}{(2\beta)_\sigma}, & \text{if } i, j \text{ are in } V_n, \\ 0, & \text{otherwise.} \end{cases}$$

Besides, we define for $i \in V$

$$\psi^{(n)}(i) = \begin{cases} \sum_{\sigma\in\bar{\mathcal{P}}_{i}^{(n)}} \frac{W_\sigma}{(2\beta)_\sigma^-}, & \text{if } i \text{ is in } V_n, \\ 1, & \text{otherwise.} \end{cases}$$

Recall the VRJP and its time-changed (Z_t) defined in the introduction.

Our main theorem is the following.

Theorem 6.2.1. *(i) The sequence $\hat{G}^{(n)}(i,j)$ converges a.s. to a finite random variable*

$$\hat{G}(i,j) = \lim_{n\to\infty} \hat{G}^{(n)}(i,j).$$

(ii) Let $\mathcal{F}_n$ be the σ-field generated by $(\beta_i)_{i\in V_n}$. For all $i \in V$, $\psi^{(n)}(i)$ is a positive $\mathcal{F}_n$-martingale. It converges a.s. to an integrable β-mesurable random variable $\psi(i)$. The random field $(\psi(i))_{i\in V}$ does not depend on the choice of the increasing sequence (V_n). Moreover, the quadratic variation of the vectorial martingale $(\psi^{(n)}(i))_{i\in V}$ is given by

$$< \psi(i), \psi(j) >_n = \hat{G}^{(n)}(i,j).$$

In particular, $\psi^{(n)}(i)$ is bounded in L^2 if and only if $\mathbb{E}(\hat{G}(i,j)) < \infty$.

(iii) Let γ be a random variable independent of the field $(\beta_j)_{j\in V}$ and with law Gamma$(\frac{1}{2}, 1)$ (that is, with density $\mathbb{1}_{\gamma>0}\frac{1}{\sqrt{\pi\gamma}}e^{-\gamma}$). Define

$$G(i,j) = \hat{G}(i,j) + \frac{1}{2}\gamma^{-1}\psi(i)\psi(j),$$

Then the time changed VRJP (Z_t) on V with conductances $(W_{i,j})$ starting from i_0, is a mixture of Markov Jump processes with jump rates from i to j

$$\frac{1}{2}W_{i,j}\frac{G(i_0,j)}{G(i_0,i)} \tag{6.2.3}$$

We denote P_x^{β,γ,i_0} the law of Markov jump process which jumps from i to j at rate (6.2.3) starting from x. Hence, it means that

$$\mathbb{P}_{i_0}^{VRJP}(\cdot) = \int P_{i_0}^{\beta,\gamma,i_0}(\cdot)\nu^{\mathcal{G},W}(d\beta)\frac{\mathbb{1}_{\gamma>0}}{\sqrt{\pi\gamma}}e^{-\gamma}d\gamma.$$

(iv) We have a.s.

- *The Markov process P^{β,γ,i_0} is transient if and only if $\psi(i) > 0$ for all $i \in V$,*

- *The Markov process P^{β,γ,i_0} is recurrent if and only if $\psi(i) = 0$ for all $i \in V$.*

Notation 6.2.1. *We denote $\nu^{\mathcal{G},W}(d\beta, d\gamma) = d\nu^{\mathcal{G},W}(d\beta) \otimes \frac{\mathbb{1}_{\gamma>0}}{\sqrt{\pi\gamma}} e^{-\gamma} d\gamma$ the joint law of (β, γ). We also set*

$$u(i, j) = \log(G(i, j)) - \log(G(i, i))$$

so that the jumping rates (6.2.3) can be expressed by

$$\frac{1}{2} W_{i,j} \frac{G(i_0, j)}{G(i_0, i)} = \frac{1}{2} W_{i,j} e^{u(i_0,j)-u(i_0,i)}.$$

Remark 6.2.2. *When the VRJP is recurrent, $G = \hat{G}$, and the VRJP can be represented by a $(\beta_j)_{j\in V}$-measurable random field. When the VRJP is transient, it is remarkable that the representation involves an extra random variable γ, which is independent of the field (β_j).*

Remark 6.2.3. *The representation (6.2.3) extends to infinite graphs the representation provided in [43], Theorem 2, for finite graphs. An interesting new feature appears in the transient regime, where the generalized eigenfunction ψ and the extra γ random variable enters the expression of $G(i, j)$. It comes out from the proof that ψ somehow corresponds to the mixing field of a VRJP starting from infinity.*

Let $\tilde{Z}_n$ be the discrete time process associated with (Z_t). Clearly it is a mixture of Markov chain, with conductances

$$W_{i,j} G(i_0, i) G(i_0, j).$$

The point (iv) of the previous theorem is in fact a consequence of a more precise assertion. Let us denote $\tau_{i_0}^+ = \inf\{n \geq 1,\ \tilde{Z}_n = i_0\}$, the first return time to i_0 by $(\tilde{Z}_n)$. We have the following proposition.

Proposition 6.2.2.

$$P_i^{\beta,\gamma,i_0}(\tau_{i_0}^+ = \infty) = \begin{cases} \frac{\psi(i_0)^2}{4\gamma\tilde{\beta}_{i_0}\hat{G}(i_0,i_0)G(i_0,i_0)} & i = i_0 \\ \frac{\psi(i_0)}{2\gamma} \frac{\hat{G}(i_0,i_0)\psi(i)-\hat{G}(i_0,i)\psi(i_0)}{\hat{G}(i_0,i_0)G(i_0,i)} & i \neq i_0 \end{cases}$$

where $\tilde{\beta}_{i_0} = \sum_{j\sim i_0} \frac{1}{2} W_{i_0,j} \frac{G(i_0,j)}{G(i_0,i_0)}$.

Using Doob's h transform, the law of the process (Z_t) conditioned on the event $\{\tau_0^+ < \infty\}$ or $\{\tau_0^+ = \infty\}$ can be computed and takes a rather nice form, both in the annealed and quenched cases. We provide these formulae in Section 6.7.

A natural question that emerges from point (iv) of the theorem is that of a 0-1 law for transience/recurrence. We do not have a general answer but we have an answer in the case of vertex transitive graphs of conductances. We say that $(\mathcal{G}, W)$ is vertex transitive if the group of automorphisms of $\mathcal{G}$ that leave invariant $(W_{i,j})$ is transitive on vertices. In particular, it is the case for the cubical graph $\mathbb{Z}^d$ with constant conductances $W_{i,j} = W$. Denote by $\mathcal{A}$ the group of automorphisms that leave invariant W.

Proposition 6.2.3. *If $(\mathcal{G}, W)$ is vertex transitive and $\mathcal{G}$ infinite, then under ν^W, β, ψ, $\hat{G}$ are stationary and ergodic for the group of transformations $\mathcal{A}$. Moreover, the VRJP is either recurrent or transient, i.e.*

$$\mathbb{P}_{i_0}^{VRJP}(\text{ every vertex is visited i.o. }) = 1, \text{ or } \mathbb{P}_{i_0}^{VRJP}(\text{ every vertex is visited f.o. }) = 1.$$

In the first case $\psi(i) = 0$ for all $i \in V$, a.s., in the second case $\psi(i) > 0$ for all $i \in V$, a.s.

N.B : The action of $\mathcal{A}$ on $\hat{G}$ is $(\tau\hat{G})(i, j) = \hat{G}(\tau i, \tau j)$ for $\tau \in \mathcal{A}$.

6.2.2 Relation with random Schrödinger operators

Let us now relate Theorem 6.2.1 to the properties of the Schrödinger operator associated with the random field (β_j). Define the operator $P = (P_{i,j})_{i,j\in V}$ by

$$P_{i,j} = \begin{cases} -W_{i,j}, & \text{if } i \sim j, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we consider the Schrödinger operator on $\mathcal{G}$

$$H_\beta = -P + 2\beta,$$

where β represents the operator of multiplication by the field (β_j).

Theorem 6.2.2. *(i) The spectrum of H_β is included in $[0, \infty)$*

(ii) The operator $\hat{G}$ is the inverse of H_β in the following sense : for all $i, j \in V$, a.s.

$$\hat{G}(i, j) = \lim_{\epsilon>0,\epsilon\to 0} (H_\beta + \epsilon)^{-1}(i, j).$$

(iii) We have $(H_\beta \psi)(i) = 0$ a.s. for all $i \in V$.

(iv) In the case of the grid $\mathbb{Z}^d$ and when $W_{i,j} = W$ is constant, $\hat{G}$ and ψ are stationary ergodic for the spacial shift. Moreover, in the transient case, ψ is a positive generalized eigenfunction with eigenvalue 0 in the sense that $H_\beta\psi = 0$ and ψ has at most polynomial growth, i.e. there exists $C > 0$ and $p \geq 0$ such that for all $i \in \mathbb{Z}^d$, a.s.

$$|\psi(i)| \leq C\|i\|^p.$$

6.2.3 Functional central limit theorem

Consider the VRJP on $\mathbb{Z}^d$, $d \geq 3$, and $W_{i,j} = W$ for all i, j. We prove a functional central limit theorem for the discrete time process $(\tilde{Z}_n)$ at weak reinforcement (i.e. for W large enough).

Theorem 6.2.3. *Consider the discrete time VRJP $(\tilde{Z}_n)_{n\geq 0}$ on $\mathbb{Z}^d$, $d \geq 3$, with constant $W_{i,j} = W$. Denote*

$$B_t^{(n)} = \frac{\tilde{Z}_{[nt]}}{\sqrt{n}}.$$

There exists $\lambda_2 > 0$ such that if $W > \lambda_2$, the discrete time VRJP $(\tilde{Z}_n)$ satisfies a functional central limit theorem, i.e. under $\mathbb{P}_0^{VRJP}$, $B_t^{(n)}$ converges in law (for the Skorokhod topology) to a d-dimensional Brownian motion B_t with non degenerate isotropic diffusion matrix $\sigma^2 Id$, for some $0 < \sigma^2 < \infty$.

6.2.4 Consequences for the Edge Reinforced Random Walk (ERRW)

The Edge Reinforced Random Walk (ERRW) is a famous discrete time process introduced in 1986 by Coppersmith and Diaconis, [16].

Endow the edges of the graph by some positive weights $(a_e)_{e\in E}$. Let $(X_n)_{n\in\mathbb{N}}$ be a random process that takes values in V, and let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$ be the filtration of its past. For any $e \in E$, $n \in \mathbb{N}$, let

$$N_n(e) = a_e + \sum_{k=1}^{n} \mathbb{1}_{\{\{X_{k-1},X_k\}=e\}} \tag{6.2.4}$$

be the number of crosses of the (non-directed) edge e up to time n plus the initial weight a_e.

Then $(X_n)_{n\in\mathbb{N}}$ is called Edge Reinforced Random Walk (ERRW) with starting point $i_0 \in V$ and weights $(a_e)_{e\in E}$, if $X_0 = i_0$ and, for all $n \in \mathbb{N}$,

$$\mathbb{P}(X_{n+1} = j \mid \mathcal{F}_n) = \mathbb{1}_{\{j\sim X_n\}} \frac{N_n(\{X_n, j\})}{\sum_{k\sim X_n} N_n(\{X_n, k\})}. \tag{6.2.5}$$

We denote by $\mathbb{P}^{ERRW}_{i_0}$ the law of the ERRW starting from the initial vertex i_0.

Important progress have been done in the last ten years in the understanding of this process, cf e.g. [4, 22, 37, 45]. In particular, in was proved in 2012 by Sabot, Tarrès, [45], and Angel, Crawford, Kozma, [4], on any graph with bounded degree at strong reinforcement (i.e. for $a_e < \tilde\lambda_0$ for some fixed $\tilde\lambda_0 > 0$) that the ERRW is a mixture of positive recurrent Markov chains. It was proved by Disertori, Sabot, Tarrès [22] that on $\mathbb{Z}^d$, $d \geq 3$, the ERRW is transient at weak reinforcement (i.e. for $a_e > \tilde\lambda_1$ for some fixed $\tilde\lambda_1 < \infty$).

From Theorem 1 of [45], we know that the ERRW has the law of a VRJP in independent conductances. More precisely, consider $(W_e)_{e\in E}$ as independent random variables with law Gamma(a_e). Consider the VRJP in conductances $(W_e)_{e\in E}$ and its underlying discrete time process $(\tilde Y_n)$. Then the annealed law of $(\tilde Y_n)$ (after expectation with respect to W) is that of the ERRW (X_n) with initial weights (a_e). Hence, we can apply Theorem 6.2.1 at fixed W. We thus consider the joint law $\tilde\nu^{\mathcal{G},a}(dW, d\beta, d\gamma)$ of W, β, γ defined for any test function F by

$$\int F(W, \beta, \gamma)\tilde\nu^{\mathcal{G},a}(dW, d\beta, d\gamma) = \mathbb{E}\left(\int F(W, \beta, \gamma)\nu^{\mathcal{G},W}(d\beta, d\gamma)\right),$$

where the expectation is with respect to the random variables (W). We simply denote by $\tilde\nu^{\mathcal{G},a}(dW, d\beta)$, $\tilde\nu^{\mathcal{G},a}(d\beta)$ the corresponding marginals. From Theorem 6.2.1 we, see that the ERRW starting from i_0 is a mixture of reversible Markov chain with conductances

$$W_{i,j}G(i_0, i)G(i_0, j), \tag{6.2.6}$$

where G is defined in Theorem 6.2.1, and (W, β, γ) are distributed according to $\tilde\nu^{\mathcal{G},a}(dW, d\beta, d\gamma)$.

One useful point is that we keep the 1-dependence of the field β, after expectation with respect to W.

Proposition 6.2.4. *Under $\tilde\nu^{\mathcal{G},a}(d\beta)$, $(\beta_j)_{j\in V}$ is 1-dependent : if $U, U' \subset V$ are such that $\mathrm{d}_{\mathcal{G}}(U, U') \geq 2$, then $(\beta_i)_{i\in U}$ and $(\beta_j)_{j\in U'}$ are independent.*

Proof. Indeed, from Proposition 6.2.1, the Laplace transform of $(\beta_i)_{i\in U}$ only involves the conductances $W_{i,j}$ for i or j in U. This implies that the joint Laplace transform of $(\beta_i)_{i\in U}$ and $(\beta_i)_{i\in U'}$ is still the product of Laplace transforms even after taking expectation with respect to the random variables (W_e). □

This yields a counterpart of Proposition 6.2.3 for the ERRW.

Proposition 6.2.5. *Assume $(\mathcal{G}, (a_{i,j}))$ is vertex transitive with automorphism group $\mathcal{A}$, and $\mathcal{G}$ infinite. Then under $\tilde\nu^{\mathcal{G},a}$, W, β, ψ, $\hat G$ are stationary and ergodic for the group of transformations $\mathcal{A}$. Moreover, the ERRW is either recurrent or transient, i.e.*

$$\mathbb{P}^{ERRW}_{i_0}(\text{ every vertex is visited i.o. }) = 1, \text{ or } \mathbb{P}^{ERRW}_{i_0}(\text{ every vertex is visited f.o. }) = 1.$$

In the first case $\psi(i) = 0$ for all $i \in V$, a.s., in the second case $\psi(i) > 0$ for all $i \in V$, a.s.

N.B : The action of $\mathcal{A}$ on $\hat G$ and W is $(\tau\hat G)(i, j) = \hat G(\tau i, \tau j)$, $\tau W_{i,j} = W_{\tau i,\tau j}$ for $\tau \in \mathcal{A}$.

Remark 6.2.4. *In [36], it was proved on infinite graphs that the ERRW is a mixture of Markov chains, obtained as a weak limit of the mixing measure of the ERRW on finite approximating graphs. The difference in the representation we give in (6.2.6) is that the random variables ψ, $\hat{G}$ are obtained as almost sure limits and hence are measurable functions of the random variables β. This yields some stationarity and ergodicity, which are the key ingredients in the 0-1 law, and in forthcoming Theorems 6.2.4 and 6.2.5.*

Remark 6.2.5. *It seems that this 0-1 law is new, both for the VRJP and the ERRW. In [36], it was proved that if the ERRW comes back with probability 1 to its starting point then it visits infinitely often all points, a.s., which is a weaker result. This was proved using the representation of the ERRW as mixture of Markov chains of [36]. A short proof of that result can also be given, cf [51].*

We now give a counterpart of Theorem 6.2.3 for the ERRW@. It is a consequence of Theorem 6.2.1 and of the delocalization result proved by Disertori, Sabot, Tarrès in [22].

Theorem 6.2.4. *Consider the ERRW $(X_n)_{n\geq 0}$ on $\mathbb{Z}^d$, $d \geq 3$, with constant weights $a_{i,j} = a$. Denote*

$$B_t^{(n)} = \frac{X_{[nt]}}{\sqrt{n}}.$$

There exists $\tilde{\lambda}_2 > 0$ such that if $a > \tilde{\lambda}_2$, the ERRW satisfies a functional central limit theorem, i.e. under $\mathbb{P}_0^{ERRW}$, $(B_t^{(n)})$ converges in law (for the Skorokhod topology) to a d-dimensional Brownian motion (B_t) with non degenerate isotropic diffusion matrix $\sigma^2 Id$, for some $0 < \sigma^2 < \infty$.

Finally, we can deduce recurrence of the ERRW in dimension 2 from the estimates obtained by Merkl Rolles in [37][1].

Theorem 6.2.5. *The ERRW $(X_n)_{n\geq 0}$ on $\mathbb{Z}^2$ with constant weights $a_{i,j} = a$ is a.s. recurrent, i.e.*

$$\mathbb{P}_0^{ERRW}\left(\text{ every vertex is visited infinitely often }\right) = 1.$$

In [37], Merkl and Rolles proved polynomial decrease of the type

$$\mathbb{E}\left(\left(\frac{x_v}{x_0}\right)^{\frac{1}{4}}\right) \leq C|v|^{-\xi}, \tag{6.2.7}$$

for some $\xi > 0$, where x_v is the conductance at the site v for the mixing measure of the ERRW, uniformly for a sequence of finite approximating graphs. When $0 < \xi < 1$, it does not give by itself enough information to prove recurrence. It was used in the case of a diluted 2-dimensional graphs to prove positive recurrent at strong reinforcement. The extra information given by the representation (6.2.6) and the stationarity of ψ, implies that the polynomial estimate (6.2.7) is incompatible with $\psi(i) > 0$ and hence is incompatible with transience. Detailed arguments are provided in Section 6.8.

6.2.5 Open questions

The most important question certainly concerns the relation between the properties of the VRJP and the spectral properties of the random Schrödinger operator H_β. We think that on $\mathbb{Z}^d$ with constant weights $W_{i,j} = W$, recurrence/transience of the VRJP is related to the localized/delocalized regimes of H_β. Let us dare a conjecture : we think that the transient regime of the VRJP coincides with the existence of extended states at least at the bottom of the spectrum of H_β. It might at first seem inconsistant to expect

[1]We are grateful to Franz Merkl and Silke Rolles for a useful discussion on that subject

extended states at the bottom of the spectrum since the Anderson model with i.i.d. potential is expected to be localized at the edges of the spectrum (which is proved in several cases). But this localization is a consequence of Lifchitz tails, and there are good reasons to expect that Lifchitz tails fail for the potential β, which is not i.i.d. but 1-dependent. Indeed, the bottom of the spectrum of H_β is 0, it does not coincide with the minimum of the support of the distribution of 2β translated by the spectrum of $-P$, as it is the case for i.i.d. potential. In fact, on a finite set, the minimum of the spectrum is reached on the set $\det(2\beta - P) = 0$ which is a set of codimension 1, hence it is "big".

Another natural question concerns the uniform integrability of the martingale $\psi^{(n)}(i)$. Let us ask a more precise question : is it true (at least for $\mathbb{Z}^d$ with constant weights) that transience of the VRJP implies that the martingale $\psi^{(n)}(i)$ is bounded in L^2? It is quite natural to expect such a property from relation (6.5.28) since $\hat{G}^{(n)}(i,i)$ appears to be the quadratic variation of $\psi^{(n)}(i)$. This would have several consequences. Firstly, it would imply that in dimension $d \geq 3$, the VRJP satisfies a functional central limit theorem as soon as the VRJP is transient, by the same argument as that of the proof of Theorem 6.2.3. It would also imply directly that the VRJP is recurrent as soon as the reversible Markov chain in conductances $(W_{i,j})$ is recurrent, if the group of automorphisms of $(\mathcal{G}, W)$ is transitive. Indeed, assume that the property is true and the VRJP is transient. By Theorem 6.2.1, the discrete time process $(\tilde{Z}_n)$ would be represented as a mixture of reversible Markov chains with conductances $W_{i,j}G(0,i)G(0,j)$. It is rather easy (cf Remark 6.6.1) to show that

$$\frac{\hat{G}(0,i)}{\hat{G}(0,0)} \leq \frac{\psi(i)}{\psi(0)}.$$

Hence, $(\tilde{Z}_n)$ is equivalently a mixture of Markov chains with conductances

$$\frac{\psi(0)^2}{G(0,0)^2} W_{i,j} G(0,i)G(0,j) \leq W_{i,j}\psi(i)\psi(j).$$

But $\psi(i)$ is stationary ergodic, if ψ is squared integrable, we would have

$$E(W_{i,j}\psi(i)\psi(j)) \leq C W_{i,j}$$

for some $C > 0$. Usual arguments imply that the Markov chain in conductance $W_{i,j}\psi(i)\psi(j)$ is recurrent if the Markov chain in conductances $(W_{i,j})$ is recurrent (cf e.g. Exercice 2.75, [33]). We arrive at a contradiction.

6.2.6 Organization of the paper

In Section 6.3, we gather results for finite graphs, in particular we recall the main results of [43]. In Section 6.4, we define the important notion of restriction with wired boundary condition and the compatibility property. Section 6.5 is the key step in the paper where the martingale property is proved. In Section 6.6, we prove Theorem 6.2.1, Propositions 6.2.2 and 6.2.3 and Theorem 6.2.2. In Section 6.7, we provide extra computations of h-transforms of the quenched and annealed VRJP. Section 6.8, we prove recurrence of ERRW in dimension 2 for all initial weights. In Section 6.9, we prove functional central limit theorems for the VRJP and the ERRW, Theorems 6.2.3 and 6.2.4.

6.3 The random potential β on finite graphs

We gather in this section several results for finite graphs.

6.3.1 The field β on finite graphs and relation to the VRJP

In this subsection we consider the case where $\mathcal{G} = (V, E)$ is a finite graph. Recall that every non oriented edge $e = \{i, j\}$ is labeled with a positive real number $W_e = W_{i,j}$. Firstly we recall theorem 1 from [43], which gives the density of β on any finite graph.

Theorem A ([43], Theorem 1). *Let $\mathcal{G} = (V, E)$ be a (W_e) weighted finite graph as above. The measure below is a probability on $(\mathbb{R}_+)^V$:*

$$\nu^{\mathcal{G},W}(d\beta) := \mathbb{1}_{H_\beta>0} \left(\frac{2}{\pi}\right)^{|V|/2} \exp(-\sum_{i\in V} \beta_i + \sum_{e\in E} W_e)\frac{d\beta_V}{\sqrt{\det H_\beta}} \tag{6.3.8}$$

with $d\beta_V = \prod_{i\in V} d\beta_i$, and where H_β is the Schrödinger operator on $\mathcal{G}$: $H_\beta = 2\beta - P$ where P is the adjacency matrix of the undirected graph $\mathcal{G}$ with weight (W_e), in other words, H_β is the matrix with coefficients

$$H_\beta(i, j) = \begin{cases} 2\beta_i, & i = j, \\ -W_{i,j}, & i \neq j,\ i \sim j, \\ 0, & otherwise. \end{cases}$$

If $(\beta_i, i \in V)$ is $\nu^{\mathcal{G},W}$ distributed, then, the Laplace transform of (β_i) is

$$\mathbb{E}(\exp(-\lambda \cdot \beta)) = \exp\left(-\sum_{i\sim j} W_{i,j}(\sqrt{(\lambda_i + 1)(\lambda_j + 1)} - 1)\right)\prod_{i\in V} \frac{1}{\sqrt{\lambda_i + 1}}. \tag{6.3.9}$$

for all $(\lambda_i) \in \mathbb{R}_+^V$.

The field β is closely related to the VRJP, as shown in the next two theorems. Consider the VRJP (Y_t) on $\mathcal{G}$ with weight $(W_{i,j})$ and initial local times 1, starting at $i_0 \in V$. In [45], it is shown that the time changed process $Z_t = Y_{D^{-1}}(t)$ (recall from (6.1.1) that $D(t) = \sum_{i\in V}(L_i^2(t) - 1)$) is a mixture of Markov jump process, more precisely:

Theorem B ([45], Theorem 2). *Assume V finite. The following measure is a probability distribution on the set $\{(u_i)_{i\in V} \in \mathbb{R}^V,\ u_{i_0} = 0\}$:*

$$Q^W_{i_0}(du) = \frac{1}{\sqrt{2\pi}^{|V|-1}} \exp\left(-\sum_{i\in V} u_i - \sum_{i\sim j} W_{i,j}(\cosh(u_i - u_j) - 1)\right)\sqrt{D(W, u)}du_{V\setminus\{i_0\}} \tag{6.3.10}$$

where $du_{V\setminus\{i_0\}} = \prod_{i\in V\setminus\{i_0\}} du_i$ and

$$D(W, u) = \sum_{T\in\mathcal{T}} \prod_{\{i,j\}\in T} W_{i,j}e^{u_i+u_j}$$

The sum is over $\mathcal{T}$, the set of spanning trees of the graph $\mathcal{G}$.

The law of the time changed VRJP (Z_t) starting at i_0 is a mixture of Markov jump processes starting at i_0, with jump rate $\frac{1}{2}W_{i,j}e^{u_j-u_i}$ from i to j, when (u_i) is distributed according to $Q^W_{i_0}(du)$.

Remark 6.3.1. *By the matrix-tree theorem, $D(W, u)$ is any diagonal minor of the $|V| \times |V|$ matrix $(m_{i,j})$ with coefficients*

$$m_{i,j} = \begin{cases} 0, & \text{if } i \nsim j,\ i \neq j \\ -W_{i,j}e^{u_i+u_j}, & \text{if } i \sim j,\ i \neq j \\ \sum_{k\in V,k\sim i} W_{i,k}e^{u_i+u_k}, & \text{if } i = j \end{cases}$$

Remark 6.3.2. *The probability measure $Q_{i_0}^W(du)$ appeared previously to [45] in a rather different context in the work of Disertori, Spencer, Zirnbauer, [23]. In particular, the fact that $Q_{i_0}^W(du)$ is a probability measure was proved there as a consequence of a Berezin identity applied to a supersymmetric extension of that measure.*

On finite graph, the random environment (u_i) of the previous theorem can be represented thanks to the Green function of the random potential $(\beta_i, i \in V)$. Let us recall Theorem 3 in [43].

Theorem C ([43], Theorem 3)**.** *Assume V finite. Let $(\beta_j)_{j\in V}$ be $\nu^{\mathcal{G},W}$ distributed and let $G = (H_\beta)^{-1}$ be the green function of the Schrödinger operator H_β. We denote*

$$e^{u(i,j)} = \frac{G(i,j)}{G(i,i)}. \tag{6.3.11}$$

For all $i_0 \in V$, we have the following properties

(i) the random field $(u(i_0,j))_{j\in V}$ has the distribution $Q_{i_0}^W$ of Theorem B,

(ii) $(u(i_0,j))_{j\in V}$ is $(\beta_j)_{j\in V\setminus\{i_0\}}$-measurable.

(iii) $G(i_0,i_0)$ is equal in law to $\frac{1}{2\gamma}$, where γ is a gamma random variable with parameter $(1/2,1)$,

(iv) $G(i_0,i_0)$ is independent of $(\beta_j)_{j\neq i_0}$, hence independent of the field $(u(i_0,j))_{j\in V}$,

(v) for all $i_0 \in V$, $i \in V$

$$\beta_i = \frac{1}{2}\sum_{j\sim i} W_{i,j} e^{u(i_0,j)-u(i_0,i)} + \frac{\mathbb{1}_{i=i_0}}{2G(i_0,i_0)}. \tag{6.3.12}$$

Remark 6.3.3. *Here we only consider the VRJP with initial local time 1, in fact, the above correspondence between β and VRJP still holds for the process starting with any positive local times $(\phi_i, i \in V)$, in such case, there is a corresponding density ν^{W,ϕ^2}, which is detailed in [43].*

The green function $G(\cdot,\cdot)$ has a representation as a path sum.

Proposition 6.3.1. *Let $\mathcal{P}_{i,j}^V$ be the collection of path in V from i to j, and $\bar{\mathcal{P}}_{i,j}^V$ be the collection of paths $\sigma = (\sigma_0 = i, \dots, \sigma_m = j)$ in V from i to j such that $\sigma_k \neq j, k = 0, \dots, m-1$. For all $(\beta_j)_{j\in V} \in \mathbb{R}^V$ such that $2\beta - P > 0$, we have*

$$G(i,j) = \sum_{\sigma\in\mathcal{P}_{i,j}^V} \frac{W_\sigma}{(2\beta)_\sigma} \tag{6.3.13}$$

and with the notation (6.3.11)

$$\exp(u(i,j)) = \sum_{\sigma\in\bar{\mathcal{P}}_{j,i}^V} \frac{W_\sigma}{(2\beta)_\sigma^-}. \tag{6.3.14}$$

Proof. Firstly we show that $\sum_{\sigma\in\mathcal{P}_{i,j}^V} \frac{W_\sigma}{(2\beta)_\sigma}$ converges. Note that $(2\beta - P) > 0$ is an M-matrix, $G = (2\beta-P)^{-1}$ is well defined and $G(i,j) > 0$ for all $i,j \in V$. Consider, for $K \geq 0$

$$G^K(i,j) = \sum_{\sigma\in\mathcal{P}_{i,j}^V, |\sigma|\leq K} \frac{W_\sigma}{(2\beta)_\sigma},$$

It can be shown by recurrence that for any $K \geq 0$, $G^K(i,j) \leq G(i,j)$.

- $K=0$, as β_i are a.s. strictly positive, for $i=j$ we have

$$G^0(i,i)=\frac{1}{2\beta_i}\le G(i,i).$$

(Indeed, $H_\beta G=\mathrm{Id}$, hence $2\beta_i G(i,i)-(PG)(i,i)=1$ which implies $2\beta_i G(i,i)\ge 1$.) If $i\neq j$, then clearly $G^0(i,j)=0\le G(i,j)$.

- For the inductive step, note that $GH_\beta=\mathrm{Id}$ gives for all i,j

$$2\beta_j G(i,j)-\sum_{l\sim j} W_{l,j}G(i,l)=\mathbb{1}_{i=j}. \tag{6.3.15}$$

If $G^K(i,j)\le G(i,j)$, then using the previous identity

$$\begin{aligned} G^{K+1}(i,j) &= \sum_{\sigma\in\mathcal{P}^V_{i,j},|\sigma|\le K+1}\frac{W_\sigma}{(2\beta)_\sigma} \\ &= \frac{\mathbb{1}_{i=j}}{2\beta_j}+\sum_{l\sim j}G^K(i,l)\frac{W_{l,j}}{2\beta_j} \\ &\le \frac{\mathbb{1}_{i=j}}{2\beta_j}+\sum_{l\sim j}\frac{W_{l,j}}{2\beta_j}G(i,l) \\ &= G(i,j). \end{aligned} \tag{6.3.16}$$

Let us define $G'(i,j)=\lim_{K\to\infty}G^K(i,j)=\sum_{\sigma\in\mathcal{P}^V_{i,j}}\frac{W_\sigma}{(2\beta)_\sigma}<\infty$. Note that H_β is a.s. positive definite, its inverse is uniquely determined, hence it is enough to check the equation $G'H_\beta=\mathrm{Id}$. Passing to the limit in the second equality of equation (6.3.16), gives

$$G'(i,j)=\frac{\mathbb{1}_{i=j}}{2\beta_j}+\sum_{l\sim j}G'(i,l)\frac{W_{l,j}}{2\beta_j}$$

which is equivalent to $G'H_\beta=1$.

For (6.3.14), note first that $\sum_{\sigma\in\bar{\mathcal{P}}^V_{j,i}}\frac{W_\sigma}{\beta^-_\sigma}\le\beta_i G(j,i)<\infty$ a.s.. A path in $\mathcal{P}^V_{j,i}$ can be cut at its first visit to i, turning it into the concatenation of a path in $\bar{\mathcal{P}}^V_{j,i}$ and a path in $\mathcal{P}^V_{i,i}$, and this operation is bijective. It implies that

$$\left(\sum_{\sigma\in\bar{\mathcal{P}}^V_{j,i}}\frac{W_\sigma}{(2\beta)^-_\sigma}\right)G(i,i)=\left(\sum_{\sigma\in\bar{\mathcal{P}}^V_{j,i}}\frac{W_\sigma}{(2\beta)^-_\sigma}\right)\left(\sum_{\sigma\in\mathcal{P}^V_{i,i}}\frac{W_\sigma}{(2\beta)_\sigma}\right)=\sum_{\sigma\in\mathcal{P}^V_{j,i}}\frac{W_\sigma}{(2\beta)_\sigma}=G(j,i)=G(i,j), \tag{6.3.17}$$

hence equation (6.3.14). □

6.3.2 A priori estimates on $e^{u(i,j)}$.

The following proposition is borrowed from [23], Lemma 3. For convenience we give a shorter proof of that estimate based on spanning trees instead of fermionic variables, following the proof of the corresponding result for the ERRW, c.f. [22], Lemma 7.

Proposition 6.3.2. *Let $\mathcal{G} = (V, E)$ be a finite graph with edge weights $(W_{i,j})$. Fix a vertex i_0. Let $\eta > 0$. If there exists a path $\sigma = (\sigma_0, \dots, \sigma_K)$ from $i \in V$ to $j \in V$ of length K such that $W_{\sigma_k,\sigma_{k+1}} \geq 2\eta$ for all $k = 0, \dots, K-1$, then*

$$\mathbb{E}(e^{\eta \cosh(u(i_0,j)-u(i_0,i))}) \leq 2^{K/2}$$

where $u(i_0, j)$ is the mixing field of the VRJP starting at i_0 defined in Theorem C.

Proof. We simply write $u(j)$ for $u(i_0, j)$ in this proof. By Theorem C, the density of $(u(i))$ on $\{(u_i)_{i\in V} \in \mathbb{R}^V,\ u(i_0) = 0\}$ is

$$\mathcal{Q}^W_{i_0}(du) = \frac{1}{\sqrt{2\pi}^{|V|-1}} \exp\Big(-\sum_i u(i) - \sum_{i\sim j} W_{i,j}(\cosh(u(i)-u(j)) - 1)\Big)\sqrt{D(W,u)}\, du_{V\setminus\{i_0\}},$$

with $du_{V\setminus\{i_0\}} = \prod_{i\neq i_0} du_i$.

Consider a path $\sigma_0 = i, \sigma_1, \dots, \sigma_K = j$ as in the statement of the proposition and assume that it is simple. We have

$$\cosh(u(i) - u(j)) \leq \sum_{k=1}^{K} \cosh(u(\sigma_{k-1}) - u(\sigma_k)). \tag{6.3.18}$$

Let $\tilde{W} = W - \eta \sum_{k=1}^{K} \mathbb{1}_{\{\sigma_{k-1},\sigma_k\}}$, (i.e. $\tilde{W}$ is equal to $W - \eta$ on the path and unchanged on the complement of the path). By assumption, we have $\tilde{W}_{i,j} > 0$ on the edges, and for all spanning tree T

$$\begin{aligned}\prod_{\{i,j\}\in T} W_{i,j} e^{u(i)+u(j)} &\leq \left(\prod_{k=1}^{K} \frac{W_{\sigma_{k-1},\sigma_k}}{W_{\sigma_{k-1},\sigma_k} - \eta}\right) \prod_{\{i,j\}\in T} \tilde{W}_{i,j} e^{u(i)+u(j)} \\ &\leq 2^K \prod_{\{i,j\}\in T} \tilde{W}_{i,j} e^{u(i)+u(j)},\end{aligned}$$

which implies

$$D(W, u) \leq 2^K D(\tilde{W}, u).$$

From (6.3.18) and the expression of $\mathcal{Q}^W_{i_0}(du)$, we deduce that

$$\exp(\eta \cosh(u(i) - u(j)))\mathcal{Q}^W_{i_0}(du) \leq 2^{K/2} \mathcal{Q}^{\tilde{W}}_{i_0}(du).$$

It implies that

$$\mathbb{E}(e^{\eta \cosh(u(i)-u(j))}) = \int e^{\eta \cosh(u(i)-u(j))} \mathcal{Q}^W_{i_0}(du) \leq 2^{K/2} \int \mathcal{Q}^{\tilde{W}}_{i_0}(du) = 2^{K/2}.$$

□

6.4 The wired boundary condition and Kolmogorov extension to infinite graph

6.4.1 Restriction with wired boundary condition

Our objective is to extend the relations between the VRJP and the β field to the case of infinite graphs. To this end, we need appropriate boundary condition, which turns out to be the wired boundary condition.

Definition 6.4.1. *Let $\mathcal{G} = (V, E)$ be a connected graph with finite degree at each site, and V_1 a strict finite subset of V. We define the restriction of $\mathcal{G}$ to V_1 with wired boundary condition as the graph $\mathcal{G}_1 = (\tilde{V}_1 = V_1 \cup \{\delta\}, E_1)$ where δ is an extra point and*

$$E_1 = \{\{i,j\} \in E, \text{ s.t. } i \in V_1, j \in V_1, i \sim j\} \cup \{\{i,\delta\}, i \in V_1 \text{ s.t. } \exists j \notin V_1, i \sim j\}.$$

If $(W_{i,j})_{\{i,j\}\in E}$ is a set of positive conductances, we define $(W^{(1)}_{i,j})_{\{i,j\}\in E_1}$ as the set of restricted conductances by

$$\begin{cases} W^{(1)}_{i,j} = W_{i,j}, & \text{if } i,j \in V_1,\ \{i,j\} \in E_1, \\ W^{(1)}_{i,\delta} = \sum_{j\notin V_1, j\sim i} W_{i,j}, & \text{if } \{i,\delta\} \in E_1, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 6.4.1. *Intuitively, this restriction corresponds to identify all points in $V \setminus V_1$ to a single point δ and to delete the edges connecting points of $V \setminus V_1$. The new weights are obtained by summing the weights of the edges identified by this procedure.*

The following lemma is fundamental and is the justification for the choice of this notion of restriction.

Lemma 6.4.1. *Let $\mathcal{G} = (V, E)$ be a finite graph with conductances $(W_{i,j})$. Let V_1 be a strict subset of V and $\mathcal{G}_1$ be its restriction with wired boundary condition. Let $(\beta_j)_{j\in V}$ be distributed according to $\nu^{\mathcal{G},W}$ (c.f. Proposition 6.2.1). Let $\beta^{(1)}$ be distributed according to $\nu^{\mathcal{G}_1,W^{(1)}}$. Then*

$$\beta_{|V_1} \overset{law}{=} \beta^{(1)}_{|V_1}.$$

Remark 6.4.2. *Note that there is no such compatibility relation with the more usual notion of restriction of graph. The wired boundary condition is fundamental and in fact will be responsible for the extra gamma random variable that appears in the representation of the VRJP on the infinite graph.*

Proof. Taking $\lambda_{|V\setminus V_1} = 0$ in Theorem A, the Laplace transform of $(\beta_i, i \in V_1)$ is

$$\mathbb{E}\left(e^{-\sum_{i\in V_1}\lambda_i\beta_i}\right) \tag{6.4.19}$$
$$= \exp\left(-\sum_{i\sim j, i,j\in V_1} W_{i,j}(\sqrt{(1+\lambda_i)(1+\lambda_j)}-1) - \sum_{i\sim j, i\in V_1, j\notin V_1}(W_{i,j}(\sqrt{1+\lambda_i}-1))\right)\prod_{i\in V_1}\frac{1}{\sqrt{1+\lambda_i}}.$$

Applying Theorem A to the graph $\mathcal{G}_1$ with $\lambda_\delta = 0$, we get

$$\mathbb{E}\left(e^{-\sum_{i\in V_1}\lambda_i\beta^{(1)}_i}\right) \tag{6.4.20}$$
$$= \exp\left(-\sum_{i\sim j, i,j\in V_1} W^{(1)}_{i,j}(\sqrt{(1+\lambda_i)(1+\lambda_j)}-1) - \sum_{i\in V_1, i\overset{\mathcal{G}_1}{\sim}\delta}(W^{(1)}_{i,\delta}(\sqrt{1+\lambda_i}-1))\right)\prod_{i\in V_1}\frac{1}{\sqrt{1+\lambda_i}}.$$

By definition of $W^{(1)}_{i,j}$, these Laplace transforms are equal. □

6.4.2 Kolmogorov extension

Let $\mathcal{G} = (V, E)$ be a connected infinite graph with finite degree at each site with conductances $(W_{i,j})$. Let $(V_n)_{n\geq 1}$ be an increasing sequence of finite strict subsets of V that exhausts V

$$\cup_n V_n = V.$$

Let $\mathcal{G}_n = (\tilde{V}_n = V_n \cup \{\delta_n\}, E_n)$ be the restriction of $\mathcal{G}$ to V_n with wired boundary condition, and $(W^{(n)})$ the restricted conductances. By construction, if $n < m$, then $(\mathcal{G}_n, W^{(n)})$ is the restriction with wired boundary condition of $(\mathcal{G}_m, W^{(m)})$. Let $\beta^{(n)}$ be the random field with distribution $\nu^{\mathcal{G}_n,W^{(n)}}$. By Lemma 6.4.1, we know that $(\beta^{(n)}_{|V_n})$ is a compatible sequence of random variables. By Kolmogorov extension theorem, there exists a random field $(\beta_j)_{j\in V}$, such that $\beta_{|V_n} \overset{law}{=} \beta^{(n)}_{|V_n}$. This immediately implies that (β) has the Laplace transform given in Proposition 6.2.1. We denote by $\nu^{\mathcal{G},W}$ its law.

Moreover, we can couple the sequence of random variables $(\beta^{(n)})$ on $V_n \cup \{\delta_n\}$, with distribution $\nu^{\mathcal{G}_n,W^{(n)}}$, with β and an extra independent gamma random variable. Indeed, let γ be a random variable with distribution Gamma$(\frac{1}{2}, 1)$, independent of (β). Define $\beta^{(n)}$ by

$$\beta^{(n)}_{|V_n} = \beta_{|V_n},\ \beta^{(n)}_{\delta_n} = \sum_{j\in V_n, j\sim\delta_n} \frac{1}{2} W_{j,\delta_n} e^{u^{(n)}(\delta_n,j)} + \gamma, \tag{6.4.21}$$

where $u^{(n)}$ is the field defined in Theorem C (Recall that $u^{(n)}(\delta_n,\cdot)$ only depends on $(\beta^{(n)})_{|V_n}$ and not on $\beta^{(n)}_{\delta_n}$). From Theorem C, it is clear that $(\beta^{(n)}_j)_{j\in\tilde{V}_n}$ follows the law $\nu^{\mathcal{G}^{(n)},W^{(n)}}$. We always consider $(\beta^{(n)})$ and (β) coupled in such way in the sequel. We denote, as in Theorem 6.2.1 iii), by $\nu^{\mathcal{G},W}(d\beta, d\gamma)$ the joint law of β and γ.

6.4.3 Definition of $G^{(n)}$ and the relation between $G^{(n)}$, $\hat{G}^{(n)}$, $\psi^{(n)}$ and γ.

Recall the definition of $\mathcal{P}^{(n)}_{i,j}$ given in Section 6.2. It is clear from the definition given in the previous section that $\mathcal{P}^{(n)}_{i,j}$ coincide with $\mathcal{P}^{V_n}_{i,j}$ defined in Proposition 6.3.1. With the previous definition it implies from the same proposition that on the set V_n we have

$$(\hat{G}^{(n)})_{|V_n\times V_n} = ((H_\beta)_{|V_n\times V_n})^{-1}. \tag{6.4.22}$$

Similarly, we clearly have that $\bar{\mathcal{P}}^{(n)}_i$ defined in Section 6.2 coincides with the set $\bar{\mathcal{P}}^{\tilde{V}_n}_{i,\delta_n}$. This implies that

$$\psi^{(n)}(i) = e^{u^{(n)}(\delta_n,i)} \tag{6.4.23}$$

when $i \in V_n$, where $u^{(n)}$ corresponds to the field defined in Theorem C from the potential $\beta^{(n)}$. (Note that $u^{(n)}(\delta_n, i)$ only depends on $\beta^{(n)}_{|V_n} = \beta_{|V_n}$ and not on the value of the potential on δ_n).

Finally, we introduce the matrix $(G^{(n)}(i,j))_{i,j\in\tilde{V}_n}$ by

$$G^{(n)} = (H^{(n)}_\beta)^{-1}.$$

where as usual $H^{(n)}_\beta = 2\beta^{(n)} - P$ is the $\tilde{V}_n \times \tilde{V}_n$ Schrödinger operator relative to the potential $\beta^{(n)}$ on the graph $\mathcal{G}_n$, as in Theorem A. From (6.3.13)

$$G^{(n)}(i,j) = \sum_{\sigma\in\mathcal{P}^{\tilde{V}_n}_{i,j}} \frac{W_\sigma}{(2\beta)_\sigma}. \tag{6.4.24}$$

It is hence immediate that for i and j in V_n,

$$\hat{G}^{(n)}(i,j) \le G^{(n)}(i,j), \tag{6.4.25}$$

since $\mathcal{P}^{(n)}_{i,j} = \mathcal{P}^{V_n}_{i,j} \subset \mathcal{P}^{\tilde{V}_n}_{i,j}$ and β are a.s. positive.

Proposition 6.4.1. *With the previous notations and with the coupling of section 6.4.2*

$$G^{(n)}(\delta_n, \delta_n) = \frac{1}{2\gamma}. \tag{6.4.26}$$

Moreover,

$$G^{(n)}(i,j) = \hat{G}^{(n)}(i,j) + \psi^{(n)}(i)\psi^{(n)}(j)G^{(n)}(\delta_n, \delta_n).$$

Proof. The first equality is a direct consequence of the special choice for the coupling (6.4.21) and of the identity (6.3.12) in Theorem C.

By Proposition 6.3.1, we find that

$$\begin{cases} G^{(n)}(i,j) = \sum_{\sigma\in\mathcal{P}_{i,j}^{\tilde{V}_n}} \frac{W_\sigma}{(2\beta)_\sigma} \\ \psi^{(n)}(i) = \frac{G^{(n)}(\delta_n, i)}{G^{(n)}(\delta_n,\delta_n)} \end{cases}$$

Therefore, if we denote $\mathcal{P}_{i,\delta_n,j}^{\tilde{V}_n}$ the collection of paths on $\tilde{V}_n$ starting from i, visiting δ_n at least once, and ending at j, that is,

$$\mathcal{P}_{i,\delta_n,j}^{\tilde{V}_n} = \{\sigma = (\sigma_0, \cdots, \sigma_m) \in \mathcal{P}_{i,j}^{\tilde{V}_n}, \text{ such that } \exists 0 \le k \le m, \sigma_k = \delta_n\}$$

then

$$\begin{aligned} G^{(n)}(i,j) - \hat{G}^{(n)}(i,j) &= \sum_{\sigma\in\mathcal{P}_{i,\delta_n,j}^{\tilde{V}_n}} \frac{W_\sigma}{(2\beta^{(n)})_\sigma} \\ &= \Big(\sum_{\sigma\in\bar{\mathcal{P}}_{i,\delta_n}^{\tilde{V}_n}} \frac{W_\sigma}{(2\beta^{(n)})_\sigma^-}\Big)\cdot\Big(\sum_{\sigma\in\mathcal{P}_{\delta_n,j}^{\tilde{V}_n}} \frac{W_\sigma}{(2\beta^{(n)})_\sigma}\Big) \\ &= \psi^{(n)}(i)G^{(n)}(\delta_n, j) = \psi^{(n)}(i)\psi^{(n)}(j)G^{(n)}(\delta_n,\delta_n). \end{aligned}$$

□

6.5 The martingale property : a Ward identity

We denote by $\mathcal{F}_n = \sigma(\beta_i, i \in V_n)$, the sigma field generated by $\{\beta_i, i \in V_n\}$. The following proposition is the key property for the main theorem.

Proposition 6.5.1. *For all n, $\psi^{(n)}$ has all its moments finite. Moreover, we have*

$$\mathbb{E}\left(\psi^{(n+1)}(i)|\mathcal{F}_n\right) = \psi^{(n)}(i), \quad \forall i \in V, \tag{6.5.27}$$

and for all $i, j \in V$,

$$\mathbb{E}\left(\psi^{(n+1)}(i)\psi^{(n+1)}(j) - \psi^{(n)}(i)\psi^{(n)}(j)|\mathcal{F}_n\right) = \mathbb{E}\left(\hat{G}^{(n+1)}(i,j) - \hat{G}^{(n)}(i,j)|\mathcal{F}_n\right). \tag{6.5.28}$$

Remark 6.5.1. *In Theorem B, by the substitution $\tilde{u}(\cdot) = u(\cdot) - \frac{\sum_{i\in V} u(i)}{|V|}$, where the new variables $\tilde{u}_{V\setminus\{i_0\}}$ are in the space $\{\sum_{i\in V}\tilde{u}(i) = 0\}$, the density becomes*

$$\tilde{\mathcal{Q}}_{i_0}^W(d\tilde{u}) = \frac{1}{\sqrt{2}^{|V|-1}} e^{\tilde{u}(i_0)} e^{-\sum_{i\sim j} W_{i,j}(\cosh(\tilde{u}(i)-\tilde{u}(j))-1)} \sqrt{D(W,\tilde{u})}\, d\tilde{u}_{V\setminus\{i_0\}}.$$

We see from this expression that $e^{\tilde{u}(i)-\tilde{u}(i_0)}\cdot\tilde{\mathcal{Q}}_{i_0}^W = \tilde{\mathcal{Q}}_i^W$, hence that $\int e^{\tilde{u}(i)-\tilde{u}(i_0)}\tilde{\mathcal{Q}}_{i_0}^W(d\tilde{u}) = 1$. Applied to $V = \tilde{V}_n$, $i_0 = \delta_n$, we get $\mathbb{E}(\psi^{(n)}(i)) = 1$ which is a particular case of (6.5.27).

To simplify notations, in the sequel, for any collection of vertices U, we denote $\beta_U = \{\beta_i, i \in U\}$ and $d\beta_U = \prod_{i\in U} d\beta_i$. The proof needs some technicalities. In order to make things more transparent, we first give a non rigorous proof, containing only the core of the argument.

6.5.1 Non rigorous derivation of the martingale property

We present below an INCORRECT proof of (6.5.27) in the case $k \in V_n$. It nevertheless gives the main ideas behind the rigorous proof given in the next section and in the last section. Define the function $g^{(n)}(\beta)$ for $(\beta_i)_{i\in\tilde V_n}$ by

$$g^{(n)}(\beta) = \left(\frac{2}{\pi}\right)^{|\tilde V_n|/2} \exp(-\sum_{i\in\tilde V_n} \beta_i + \sum_{e\in E_n} W_e)\frac{1}{\sqrt{\det H_\beta^{(n)}}} \tag{6.5.29}$$

so that $\mathbb{1}_{H_\beta^{(n)}>0} g^{(n)}(\beta)d\beta = \nu^{\mathcal{G}_n, W^{(n)}}(d\beta)$.

Note that $\mathcal{G}_n$ with conductances $(W^{(n)})$, is the restriction with wired boundary condition of $(\mathcal{G}_{n+1}, W^{(n+1)})$. Lemma 6.4.1 tells us that $\beta^{(n+1)}_{|V_n} \overset{law}{=} \beta^{(n)}_{|V_n}$, which written in terms of density is

$$\int_{H_\beta^{(n+1)}>0} g^{(n+1)}(\beta_{\tilde V_{n+1}})d\beta_{\tilde V_{n+1}\setminus V_n} = \int_{H_\beta^{(n)}>0} g^{(n)}(\beta_{\tilde V_n})d\beta_{\delta_n}. \tag{6.5.30}$$

The trick is to differentiate the equality in $W^{(n+1)}_{\delta_{n+1},k}$. We can always assume that $k \sim \delta_{n+1}$: indeed, if it is not the case, we can add the edge $\{\delta_{n+1}, k\}$ "artificially", take the differential and then take the limit when $W^{(n+1)}_{\delta_{n+1},k}$ tends to 0. Note that differentiate an entry of a symmetric matrix corresponds to take two times the cofactor, therefore

$$\partial_{W^{(n+1)}_{\delta_{n+1},k}}\left(\frac{1}{\sqrt{\det H_\beta^{(n+1)}}}\right) = \frac{1}{\sqrt{\det H_\beta^{(n+1)}}} G^{(n+1)}(\delta_{n+1}, k),$$
$$\partial_{W^{(n+1)}_{\delta_{n+1},k}} g^{(n+1)}(\beta) = g^{(n+1)}(\beta)\left(G^{(n+1)}(\delta_{n+1}, k) + 1\right).$$

Therefore differentiating the left hand side of (6.5.30), we get

$$\begin{aligned}&\partial_{W^{(n+1)}_{\delta_{n+1},k}} \int_{H_\beta^{(n+1)}>0} g^{(n+1)}(\beta_{\tilde V_{n+1}})d\beta_{\tilde V_{n+1}\setminus V_n}\\ =& \int_{H_\beta^{(n+1)}>0} g^{(n+1)}(\beta_{\tilde V_{n+1}})\left(G^{(n+1)}(\delta_{n+1}, k) + 1\right) d\beta_{\tilde V_{n+1}\setminus V_n}\end{aligned} \tag{6.5.31}$$

We always consider $W^{(n)}$ as a function of $W^{(n+1)}$, obtained by the operation of wired boundary condition, Definition 6.4.1. Thus, the differentiation with respect to $W^{(n+1)}_{\delta_{n+1},k}$, amounts to a differential with respect to $W^{(n)}_{\delta_n,k}$, when $k \in V_n$. Differentiating both sides of (6.5.30) with respect to $W^{(n+1)}_{\delta_{n+1},k}$ and $W^{(n)}_{\delta_n,k}$, the same computation as (6.5.31) for the graph $\mathcal{G}_n$, gives the following equality

$$\begin{aligned}&\int_{H_\beta^{(n+1)}>0} g^{(n+1)}(\beta_{\tilde V_{n+1}})\left(G^{(n+1)}(\delta_{n+1}, k) + 1\right) d\beta_{\tilde V_{n+1}\setminus V_n}\\ =& \int_{H_\beta^{(n)}>0} g^{(n)}(\beta_{\tilde V_n})\left(G^{(n)}(\delta_n, k) + 1\right) d\beta_{\delta_n}.\end{aligned} \tag{6.5.32}$$

Note that by (6.5.30), the 1's of both sides simplify, then dividing both terms by the left/right hand side of the equality (6.5.30), it gives

$$\mathbb{E}\left(G^{(n+1)}(\delta_{n+1},k) \mid \mathcal{F}_n\right) = \mathbb{E}\left(G^{(n)}(\delta_n,k) \mid \mathcal{F}_n\right).$$

We now write $G^{(n+1)}(\delta_{n+1},k) = G^{(n+1)}(\delta_{n+1},\delta_{n+1})\psi^{(n+1)}(k)$ and $G^{(n)}(\delta_n,k) = G^{(n)}(\delta_n,\delta_n)\psi^{(n)}(k)$. Since $G^{(n+1)}(\delta_{n+1},\delta_{n+1})$ and $G^{(n)}(\delta_n,\delta_n))$ are independent of $(\beta_i)_{i\in V_n}$ it gives

$$\begin{aligned} & \mathbb{E}\left(G^{(n+1)}(\delta_{n+1},\delta_{n+1})\right)\mathbb{E}\left(\psi^{(n+1)}(k) \mid \mathcal{F}_n\right) \\ = \; & \mathbb{E}\left(G^{(n)}(\delta_n,\delta_n)\right)\mathbb{E}\left(\psi^{(n)}(k) \mid \mathcal{F}_n\right). \end{aligned} \tag{6.5.33}$$

Since $G^{(n+1)}(\delta_{n+1},\delta_{n+1})$ and $G^{(n)}(\delta_n,\delta_n)$ have the same law as $\frac{1}{2\gamma}$, we can simplify and it gives the equality.

The proof presented above is not correct for the following two reasons

- The domain of integration in (6.5.31) depends on $W^{(n+1)}_{\delta_{n+1},k}$. The derivation with respect to $W^{(n+1)}_{\delta_{n+1},k}$ should give a contribution on the boundary $\det H^{(n+1)}_\beta = 0$ on which the integrand explodes.
- In (6.5.33), $G^{(n+1)}(\delta_{n+1},\delta_{n+1})$ has the law of $\frac{1}{2\gamma}$ which is not integrable (and idem for $G^{(n)}(\delta_n,\delta_n)$).

The same trick solves the two problems : since $G^{(n)}(\delta_n,\delta_n)$ is independent of $(\beta_i)_{i\in V_n}$ and hence of $(\psi^{(n)})$, we can multiply the density by a function $e^{-\Psi(G^{(n)}(\delta_n,\delta_n))}$ such that $\mathbb{E}(e^{-\Psi(G^{(n)}(\delta_n,\delta_n))}) = 1$. Take e.g. $\Psi(x) = x/2-1$. Under the distribution $e^{-\Psi(G^{(n)}(\delta_n,\delta_n))}\nu^{\mathcal{G}_n,W^{(n)}}$, $G^{(n)}(\delta_n,\delta_n)$ is integrable and the new density vanishes on a subset of the boundary of codimension 1 (indeed, when $\det H^{(n)}_\beta = 0$, then $G^{(n)}(\delta_n,\delta_n) = +\infty$ unless the minor obtained on $V_n\times V_n$ is also zero).

6.5.2 Martingale property : the rigorous proof

The fact that $\psi^{(n)}(i)$ has all its moments finite follows easily from Proposition 6.3.2 since $\psi^{(n)}(i) = e^{u^{(n)}(\delta_n,i)}$ when $i\in V_n$, cf (6.4.23), or $\psi^{(n)}(i) = 1$ when $i\notin V_n$.

The rigorous proof of the martingale property follows the strategy described above. Consider the function

$$\Psi(x) = x/2-1. \tag{6.5.34}$$

Note that with this choice we have

$$\mathbb{E}(e^{-\Psi(\frac{1}{2\gamma})}) = 1. \tag{6.5.35}$$

Indeed γ is a gamma random variable with parameter $1/2$. Changing to variable $x = \frac{1}{2\gamma}$, the expectation equals $\int_0^\infty \sqrt{\frac{1}{2\pi x^3}}e^{-\frac{1}{2}x-\frac{1}{2x}+1}dx$. The integrand is the density of the inverse gaussian random variable with parameters $(1,1)$, hence has integral 1.

In this section we will simply write $\nu^{(n)}$ for $\nu^{\mathcal{G}_n,W^{(n)}}$. Consider the function

$$f^{(n)}(\beta_{\tilde V_n}) := \left(\frac{2}{\pi}\right)^{|\tilde V_n|/2}\frac{\exp(-\sum_{i\in\tilde V_n}\beta_i + \sum_{\{i,j\}\in E_n}W_{i,j} - \Psi(G(\delta_n,\delta_n)))}{\sqrt{\det H^{(n)}_\beta}}. \tag{6.5.36}$$

$$= e^{-\Psi(G(\delta_n,\delta_n))}g^{(n)}(\beta). \tag{6.5.37}$$

where $g^{(n)}$ is defined in (6.5.29). Then

$$\hat{\nu}^{(n)}(d\beta) := \mathbb{1}_{H_\beta^{(n)}>0}\, f^{(n)}(\beta)d\beta = e^{-\Psi(G(\delta_n,\delta_n))}\nu^{(n)}(d\beta).$$

is a probability distribution supported on the set $\{(\beta_j)_{j\in\tilde{V}_n},\quad 2\beta - P > 0\}$. We will write $\mathbb{E}_{\hat{\nu}^{(n)}}$ for the expectation under the law $\hat{\nu}^{(n)}$, we simply write $\mathbb{E}$ when the expectation is under $\nu^{(n)}$, the usual law of $\beta^{(n)}$. Since $G^{(n)}(\delta_n,\delta_n)$ is independent of $(\beta_j)_{j\in V_n}$, it is also independent under $\hat{\nu}^{(n)}$ and moreover $\beta^{(n)}_{|V_n}$ has the same law under $\nu^{(n)}$ and $\hat{\nu}^{(n)}$. This implies that we still have the relation

$$\beta^{(n)}_{|V_n} \text{ under } \hat{\nu}^{(n)} \overset{law}{=} \beta^{(n+1)}_{|V_n} \text{ under } \hat{\nu}^{(n+1)}.$$

Expressed in terms of marginal densities it gives the following lemma.

Lemma 6.5.1. *The following holds*

$$\int \mathbb{1}_{H_\beta^{(n+1)}>0}\, f^{(n+1)}(\beta_{\tilde{V}_{n+1}})d\beta_{\tilde{V}_{n+1}\setminus V_n} = \int \mathbb{1}_{H_\beta^{(n)}>0}\, f^{(n)}(\beta_{\tilde{V}_n})d\beta_{\delta_n}. \tag{6.5.38}$$

Remark 6.5.2. *The specific choice of the function Ψ in (6.5.34) is somehow irrelevant : as it will appear in the sequel, we could take in (6.5.34) any function Ψ such that (6.5.35) holds, and such that $\mathbb{E}(\frac{1}{\gamma}e^{-\Psi(\gamma)}) < \infty$.*

Proof of Proposition 6.5.1 first part: (6.5.27). Both parts of the proposition comes from differentiating (6.5.38) with respect to $W_{i,j}^{(n+1)}$ for i,j in $\tilde{V}_{n+1}$. We can always assume that $i \sim j$, as if it is not the case, we can add an 'artificial' edge $\{i,j\}$, take the differential and then let $W_{i,j}$ go to 0. We always consider $W^{(n)}$ as a function of $W^{(n+1)}$ as its restriction with wired boundary condition, cf Definition 6.4.1. It is easy to see that differentiation with respect to $W^{(n+1)}$ translates into differentiation with respect to $W^{(n)}$ as follows :

$$\begin{cases} \frac{\partial}{\partial W_{i,j}^{(n+1)}} \longrightarrow \frac{\partial}{\partial W_{i,j}^{(n)}}, & \text{if } i,j \in V_n, \\ \frac{\partial}{\partial W_{i,j}^{(n+1)}} \longrightarrow \frac{\partial}{\partial W_{\delta_n,j}^{(n)}}, & \text{if } i \in \tilde{V}_{n+1}\setminus V_n,\ j\in V_n, \\ \frac{\partial}{\partial W_{i,j}^{(n+1)}} \longrightarrow 0, & \text{if } i,j \in \tilde{V}_{n+1}\setminus V_n, \end{cases} \tag{6.5.39}$$

N.B. : more formally, if $g(W^{(n)})$ is a function of $W^{(n)}$, hence a function of $W^{(n+1)}$ since $W^{(n)}$ is considered as a function of $W^{(n+1)}$, then in the first case if means that $\frac{\partial}{\partial W_{i,j}^{(n+1)}}g(W^{(n)}) = (\frac{\partial}{\partial W_{i,j}^{(n)}}g)(W^{(n)})$.

In the sequel, we will frequently drop the superscript in $W^{(n)}$ or $W^{(n+1)}$ when no ambiguity is possible. We need first to invert the differential and the integral. Some care must be taken, first of all we need to dominate the integrand by an integrable function. Secondly, the domain of integration depends on W, this can be handled by the technique of "dérivée particulaire"; in short, it turns out that the boundary of the domain has no effect to the differentiation. The proof of these technical details are left in the last section, we recapitulate these technical issue in the following lemma:

Lemma 6.5.2. *We have : for any $i,j \in \tilde{V}_{n+1}$*

$$\partial_{W_{i,j}} \int_{H_\beta^{(n+1)}>0} f^{(n+1)}(\beta_{\tilde{V}_{n+1}})d\beta_{\tilde{V}_{n+1}\setminus V_n} = \int_{H_\beta^{(n+1)}>0} \partial_{W_{i,j}} f^{(n+1)}(\beta_{\tilde{V}_{n+1}})d\beta_{\tilde{V}_{n+1}\setminus V_n}. \tag{6.5.40}$$

Similarly, for $i,j\in\tilde{V}_n$, we have

$$\partial_{W_{i,j}} \int_{H_\beta^{(n)}>0} f^{(n)}(\beta_{\tilde{V}_n})d\beta_{\delta_n} = \int_{H_\beta^{(n)}>0} \partial_{W_{i,j}} f^{(n)}(\beta_{\tilde{V}_n})d\beta_{\delta_n}. \tag{6.5.41}$$

Remark 6.5.3. *By considering $V_n = \emptyset$ in (6.5.40) we obtain*

$$\partial_{W_{i,j}} \int_{H_\beta^{(n)}>0} f^{(n)}(\beta_{\tilde{V}_n}) d\beta_{\tilde{V}_n} = \int_{H_\beta^{(n)}>0} \partial_{W_{i,j}} f^{(n)}(\beta_{\tilde{V}_n}) d\beta_{\tilde{V}_n}. \tag{6.5.42}$$

Remark 6.5.4. *Let us explain intuitively that result. From general results, under assumption on the regularity of the integrand and of the boundary (cf e.g. [50] p. 18), the derivative (6.5.41) should be equal to the integral on the right hand side plus an extra integral on the boundary of the set that takes into account the dependence of the domain as a function of the parameters. Here, the boundary set equals $\{\beta,\ \det(H_\beta) = 0\}$. On this set the function $f^{(n)}$ vanishes almost surely. Indeed, when $\det(H_\beta^{(n)}) = 0$, then $G^{(n)}(\delta_n, \delta_n) = +\infty$ unless the principal minor obtained by removing the line and column δ_n is zero. Hence $\Psi(G^{(n)}(\delta_n, \delta_n)) = \infty$ almost surely on the boundary. This kills the boundary term. The general formula given e.g. in [50], does not apply directly, hence in the last section we adapt the proof to our context.*

We first prove (6.5.27) when $k \in V_n$. From previous considerations and Lemma 6.5.2, we get from (6.5.38) when $k \in V_n$

$$\int_{H_\beta^{(n+1)}>0} \frac{\partial}{\partial W_{\delta_{n+1,k}}} f^{(n+1)}(\beta_{\tilde{V}_{n+1}}) d\beta_{\tilde{V}_{n+1}\setminus V_n} = \int_{H_\beta^{(n+1)}>0} \frac{\partial}{\partial W_{\delta_{n,k}}} f^{(n)}(\beta_{\tilde{V}_n}) d\beta_{\delta_n}. \tag{6.5.43}$$

Note that differentiate an entry of a symmetric matrix corresponds to take two times the cofactor, therefore

$$\begin{aligned}
&\partial_{W_{\delta_n,k}}\Big(\frac{1}{\sqrt{\det H_\beta^{(n)}}}\Big) = \frac{1}{\sqrt{\det H_\beta^{(n)}}} G^{(n)}(\delta_n, k),\\
&\partial_{W_{\delta_n,k}} G^{(n)}(\delta_n, \delta_n) = 2G^{(n)}(\delta_n, \delta_n) G^{(n)}(\delta_n, k),\\
&\partial_{W_{\delta_n,k}} (e^{-\Psi(G^{(n)}(\delta_n,\delta_n))}) = -2e^{-\Psi(G^{(n)}(\delta_n,\delta_n))}\Psi'(G^{(n)}(\delta_n, \delta_n))G^{(n)}(\delta_n, \delta_n)G^{(n)}(\delta_n, k).
\end{aligned}$$

Denote

$$\Xi^{(n)}(k) = 1 + G^{(n)}(\delta_n, k) - 2\Psi'(G^{(n)}(\delta_n, \delta_n))G^{(n)}(\delta_n, \delta_n)G^{(n)}(\delta_n, k)$$

From (6.5.43) we get

$$\int_{H_\beta^{(n+1)}>0} \Xi^{(n+1)}(k) f^{(n+1)}(\beta_{\tilde{V}_{n+1}}) d\beta_{\tilde{V}_{n+1}\setminus V_n} = \int_{H_\beta^{(n)}>0} \Xi^{(n)}(k) f^{(n)}(\beta_{\tilde{V}_n} d\beta_{\delta_n} \tag{6.5.44}$$

By (6.5.38), the 1's in the $\Xi^{(\cdot)}$ of both sides simplify. Moreover, as $\psi^{(n)}(k) = \dfrac{G^{(n)}(\delta_n, k)}{G^{(n)}(\delta_n, \delta_n)}$, denoting $\tilde{\Psi}(x) = x(1 - 2x\Psi'(x))$, we have

$$\begin{aligned}
&\int_{H_\beta^{(n+1)}>0} \tilde{\Psi}(G^{(n+1)}(\delta_{n+1}, \delta_{n+1}))\psi^{(n+1)}(k) f^{(n+1)}(\beta_{\tilde{V}_{n+1}}) d\beta_{\tilde{V}_{n+1}\setminus V_n}\\
&= \int_{H_\beta^{(n)}>0} \tilde{\Psi}(G^{(n)}(\delta_n, \delta_n))\psi^{(n)}(k) f^{(n)}(\beta_{\tilde{V}_n}) d\beta_{\delta_n}
\end{aligned} \tag{6.5.45}$$

From (6.5.38), the conditional density of $(\beta_i, i \in \tilde{V}_{n+1} \setminus V_n)$ under $\hat{\nu}^{(n)}(d\beta)$, conditionally on $(\beta_i, i \in V_n)$, writes

$$\mathbb{1}_{H_\beta^{(n+1)}>0} f^{(n+1)}(\beta_{\tilde{V}_{n+1}\setminus V_n} | \beta_{V_n}) = \mathbb{1}_{H_\beta^{(n+1)}>0} \frac{f^{(n+1)}(\beta_{\tilde{V}_{n+1}})}{\int_{H_\beta^{(n)}>0} f^{(n)}(\beta_{\tilde{V}_n}) d\beta_{\delta_n}} \tag{6.5.46}$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}_{\hat{\nu}^{(n+1)}}\left(\tilde{\Psi}(G^{(n+1)}(\delta_{n+1},\delta_{n+1}))\psi^{(n+1)}(k)|\mathcal{F}_n\right)\\
&=\frac{1}{\int_{H_\beta^{(n)}>0} f^{(n)}(\beta_{\tilde{V}_n})d\beta_{\delta_n}}\int_{H_\beta^{(n+1)}>0}\tilde{\Psi}(G^{(n+1)}(\delta_{n+1},\delta_{n+1}))\psi^{(n+1)}(k)f^{(n+1)}(\beta_{\tilde{V}_{n+1}})d\beta_{\tilde{V}_{n+1}\setminus V_n}\\
&=\frac{1}{\int_{H_\beta^{(n)}>0} f^{(n)}(\beta_{\tilde{V}_n})d\beta_{\delta_n}}\int_{H_\beta^{(n)}>0}\tilde{\Psi}(G^{(n)}(\delta_n,\delta_n))\psi^{(n)}(k)f^{(n)}(\beta_{\tilde{V}_n})d\beta_{\delta_n}\\
&=\mathbb{E}_{\hat{\nu}^{(n)}}\left(\tilde{\Psi}(G^{(n)}(\delta_n,\delta_n))\psi^{(n)}(k)|\mathcal{F}_n\right),
\end{aligned}
\tag{6.5.47}
$$

where we used (6.5.45) in the third equality. As remarked before, under $\hat{\nu}^{(n)}(d\beta)$, $G^{(n)}(\delta_n,\delta_n)$ is independent of β_{V_n}, and β_{V_n} has the same law under $\nu^{(n)}$ and $\hat{\nu}^{(n)}$. Hence, the previous equality is equivalent to

$$
\begin{aligned}
&\mathbb{E}(e^{-\Psi(G^{(n+1)}(\delta_{n+1},\delta_{n+1})}\tilde{\Psi}(G^{(n+1)}(\delta_{n+1},\delta_{n+1})))\mathbb{E}\left(\psi^{(n+1)}(k)|\mathcal{F}_n\right)\\
&=\mathbb{E}(e^{-\Psi(G^{(n)}(\delta_n,\delta_n)}\tilde{\Psi}(G^{(n)}(\delta_n,\delta_n)))\mathbb{E}\left(\psi^{(n)}(k)|\mathcal{F}_n\right)
\end{aligned}
$$

Since by (6.4.26)

$$
\begin{aligned}
\mathbb{E}(e^{-\Psi(G^{(n+1)}(\delta_{n+1},\delta_{n+1})}\tilde{\Psi}(G^{(n+1)}(\delta_{n+1},\delta_{n+1}))) &= \mathbb{E}(e^{-\Psi(G^{(n)}(\delta_n,\delta_n)}\tilde{\Psi}(G^{(n)}(\delta_n,\delta_n)))\\
&= \mathbb{E}(e^{-\Psi(\frac{1}{2\gamma})}\tilde{\Psi}(\frac{1}{2\gamma}))<+\infty
\end{aligned}
$$

and since $\psi^{(n)}$ is $\mathcal{F}_n$-measurable, we get

$$
\mathbb{E}\left(\psi^{(n+1)}(k)|\mathcal{F}_n\right)=\psi^{(n)}(k).
$$

Now let us turn to the case $k\in V_{n+1}\setminus V_n$ (the case $k\notin V_{n+1}$ is trivial). Since $W^{(n)}$ does not depend on $W^{(n+1)}_{\delta_{n+1},k}$ (cf (6.5.39)), we have as a counter part of (6.5.44):

$$
\int_{H_\beta^{(n+1)}>0}\Xi^{(n+1)}(k)f^{(n+1)}(\beta_{\tilde{V}_{n+1}})d\beta_{\tilde{V}_{n+1}\setminus V_n}=0 \tag{6.5.48}
$$

hence

$$
\begin{aligned}
&\int_{H_\beta^{(n+1)}>0}\tilde{\Psi}(G^{(n+1)}(\delta_{n+1},\delta_{n+1}))\psi^{(n+1)}(k)f^{(n+1)}(\beta_{\tilde{V}_{n+1}})d\beta_{\tilde{V}_{n+1}\setminus V_n}\\
&= -\int_{H_\beta^{(n+1)}>0}f^{(n+1)}(\beta_{\tilde{V}_{n+1}})d\beta_{\tilde{V}_{n+1}\setminus V_n}
\end{aligned}
\tag{6.5.49}
$$

This implies,

$$
\begin{aligned}
&\mathbb{E}_{\hat{\nu}^{(n+1)}}\left(\tilde{\Psi}(G^{(n+1)}(\delta_{n+1},\delta_{n+1}))\psi^{(n+1)}(k)|\mathcal{F}_n\right)\\
&=\frac{\int_{H_\beta^{(n+1)}>0}\tilde{\Psi}(G^{(n+1)}(\delta_{n+1},\delta_{n+1}))\psi^{(n+1)}(k)f^{(n+1)}(\beta_{\tilde{V}_{n+1}})d\beta_{\tilde{V}_{n+1}\setminus V_n}}{\int_{H_\beta^{(n+1)}>0}f^{(n+1)}(\beta_{\tilde{V}_{n+1}})d\beta_{\tilde{V}_{n+1}\setminus V_n}}\\
&=-1
\end{aligned}
$$

That is, with the same argument as before,

$$
\mathbb{E}\left(\tilde{\Psi}(G^{(n+1)}(\delta_{n+1},\delta_{n+1}))e^{-\Psi(G^{(n+1)}(\delta_{n+1},\delta_{n+1}))}\right)\mathbb{E}\left(\psi^{(n+1)}(k)|\mathcal{F}_n\right)=-1
$$

This entails that $\mathbb{E}^{\mathcal{G}_{n+1}}(\psi^{(n+1)})(k) = 1$ since

$$
\begin{aligned}
&\mathbb{E}(\tilde{\Psi}(G^{(n+1)}(\delta_{n+1},\delta_{n+1}))e^{-\Psi(G^{(n+1)}(\delta_{n+1},\delta_{n+1}))})\\
&= \mathbb{E}(\tilde{\Psi}(\frac{1}{2\gamma})e^{-\Psi(\frac{1}{2\gamma})})\\
&= \int_0^\infty \tilde{\Psi}(x)\frac{1}{\sqrt{2\pi x^3}}e^{-\Psi(x)-\frac{1}{2x}}dx\\
&= \int_0^\infty \frac{1}{\sqrt{2\pi x}}e^{-\Psi(x)-\frac{1}{2x}}dx - \int_0^\infty \sqrt{\frac{2x}{\pi}}\Psi'(x)e^{-\Psi(x)-\frac{1}{2x}}dx\\
&= -\int_0^\infty \sqrt{\frac{2x}{\pi}}e^{-\Psi(x)-\frac{1}{2x}}(\frac{1}{2x^2}-\Psi'(x))dx - \int_0^\infty \sqrt{\frac{2x}{\pi}}\Psi'(x)e^{-\Psi(x)-\frac{1}{2x}}dx\\
&= -\int_0^\infty \sqrt{\frac{1}{2\pi x^3}}e^{-\Psi(x)-\frac{1}{2x}}dx = -\mathbb{E}(e^{-\Psi(\frac{1}{2\gamma})}) = -1
\end{aligned}
\tag{6.5.50}
$$

where in the fourth equality we have used integration by part. Remark that this computation does not depend on the specific choice of Ψ but just on the fact that all the integrals are finite, which justifies Remark 6.5.2. □

Proof of Proposition 6.5.1 second part: (6.5.28). This time we differentiate (6.5.38) w.r.t. $W_{i,j}$, we first consider the case $i,j \in V_n$, again we have, using Proposition 6.4.1 in the second equality,

$$
\begin{aligned}
\partial_{W_{i,j}}(\frac{1}{\sqrt{\det H_\beta^{(n)}}}) &= \frac{1}{\sqrt{\det H_\beta^{(n)}}}G^{(n)}(i,j)\\
&= \frac{1}{\sqrt{\det H_\beta^{(n)}}}\left(\hat{G}^{(n)}(i,j) + G^{(n)}(\delta_n,\delta_n)\psi^{(n)}(i)\psi^{(n)}(j)\right).
\end{aligned}
$$

For a matrix H, denote $|H|_{i,j}$ its (i,j) minor, that is, the determinant of the submatrix formed by deleting the i-th row and j-th column, also denote $|H|_{i,j;k,l}$ the second order minor corresponding to delete the i,k-th row and j,l-th column.

$$
\begin{aligned}
\partial_{W_{i,j}}(G^{(n)}(\delta_n,\delta_n)) &= \partial_{W_{i,j}}\frac{|H_\beta^{(n)}|_{\delta_n,\delta_n}}{|H_\beta^{(n)}|}\\
&= \frac{-2|H_\beta^{(n)}|_{i,j;\delta_n,\delta_n}|H_\beta^{(n)}| + 2|H_\beta^{(n)}|_{\delta_n,\delta_n}|H_\beta^{(n)}|_{i,j}}{|H_\beta^{(n)}|}\\
&= -2G^{(n)}(\delta_n,\delta_n)(\hat{G}^{(n)}(i,j) - G^{(n)}(i,j))\\
&= 2G^{(n)}(\delta_n,\delta_n)^2\psi^{(n)}(i)\psi^{(n)}(j).
\end{aligned}
$$

Therefore,

$$
\partial_{W_{i,j}}(e^{-\Psi(G^{(n)}(\delta_n,\delta_n))}) = -2e^{-\Psi(G^{(n)}(\delta_n,\delta_n))}\Psi'(G^{(n)}(\delta_n,\delta_n))G^{(n)}(\delta_n,\delta_n)^2\psi^{(n)}(i)\psi^{(n)}(j).
$$

Let $\Xi^{(n)}(i,j)$ be

$$
\Xi^{(n)}(i,j) = 1 + \hat{G}^{(n)}(i,j) + \psi^{(n)}(i)\psi^{(n)}(j)\tilde{\Psi}(G^{(n)}(\delta_n,\delta_n))
$$

By differentiating w.r.t. $W_{i,j}$, we have a counter part of (6.5.44), which is

$$\int_{H_\beta^{(n+1)}>0} \Xi^{(n+1)}(i,j)f^{(n+1)}(\beta_{\tilde V_{n+1}})d\beta_{\tilde V_{n+1}\setminus V_n} = \int_{H_\beta^{(n)}>0} \Xi^{(n)}(i,j)f^{(n)}(\beta_{\tilde V_n})d\beta_{\delta_n}. \tag{6.5.51}$$

The same argument as in (6.5.47) leads to

$$\begin{aligned}&\mathbb{E}_{\hat\nu^{(n+1)}}(\hat G^{(n+1)}(i,j)+\psi^{(n+1)}(i)\psi^{(n+1)}(j)\tilde\Psi(G^{(n+1)}(\delta_{n+1},\delta_{n+1}))|\mathcal{F}_n)\\ &= \mathbb{E}_{\hat\nu^{(n)}}(\hat G^{(n)}(i,j)+\psi^{(n)}(i)\psi^{(n)}(j)\tilde\Psi(G^{(n)}(\delta_n,\delta_n))|\mathcal{F}_n).\end{aligned}$$

By same type of arguments as before, since $\hat G^{(n)}$ is independent of $G^{(n)}(\delta_n,\delta_n)$ under $\hat\nu^{(n)}$, and since $\hat G^{(n)}$ has the same law under $\hat\nu^{(n)}$ and $\nu^{(n)}$,

$$\begin{aligned}&\mathbb{E}(\hat G^{(n+1)}(i,j)-\hat G^{(n)}(i,j)|\mathcal{F}_n)\\ &= -\mathbb{E}(\tilde\Psi(\frac{1}{2\gamma})e^{-\Psi(\frac{1}{2\gamma})})\mathbb{E}(\psi^{(n+1)}(i)\psi^{(n+1)}(j))-\psi^{(n)}(i)\psi^{(n)}(j)|\mathcal{F}_n).\end{aligned}$$

By (6.5.50) we have the equality (6.5.28).

Turning to other cases, in a similar manner, if $i,j\in V_{n+1}\setminus V_n$, we have

$$\hat G^{(n)}(i,j)=0,\ \psi^{(n)}(i)=\psi^{(n)}(j)=1$$

and the corresponding derivation gives

$$\mathbb{E}_{\hat\nu^{(n+1)}}(\hat G^{(n+1)}(i,j)+\psi^{(n+1)}(i)\psi^{(n+1)}(j)\tilde\Psi(G^{(n+1)}(\delta_{n+1},\delta_{n+1}))|\mathcal{F}_n)=-1.$$

which gives the result by (6.5.50).

If $i\in V_n$, $j\notin V_n$, differentiation with respect to $W_{i,j}^{(n)}$ on the left hand side of (6.5.38) corresponds to differentiation with respect to $W_{\delta_n,i}^{(n)}$ on the right hand side of (6.5.38) (cf (6.5.39)). It gives in this case

$$\begin{aligned}&\mathbb{E}(\hat G^{(n+1)}(i,j)+\psi^{(n+1)}(i)\psi^{(n+1)}(j)\tilde\Psi(G^{(n+1)}(\delta_{n+1},\delta_{n+1}))|\mathcal{F}_n)\\ &= \mathbb{E}(\tilde\Psi(G^{(n)}(\delta_n,\delta_n))\psi^{(n)}(i)|\mathcal{F}_n)\end{aligned}$$

We conclude using (6.5.50) and that

$$G^{(n)}(i,j)=0,\ \psi^{(n)}(j)=1.$$

□

6.6 Passing to the limit : proof of Theorem 6.2.1, Proposition 6.2.2, Proposition 6.2.3

6.6.1 Representation by sums of paths : proof of Theorem 6.2.1 i) and ii)

Proof of Theorem 6.2.1 i). Recall the definition of $G^{(n)}$ from section 6.4.3. By Proposition 6.3.1, $G^{(n)}(i,j)$ is a.s. finite, hence $\hat G^{(n)}(i,j)$ is also a.s. finite since $\hat G^{(n)}(i,j)\le G^{(n)}(i,j)$ when i,j are in V_n, cf (6.4.25). The sequence V_n is increasing, hence $\hat G^{(n)}(i,j)$ is an increasing function of n, to prove Theorem 6.2.1 i), it is enough to show that $\hat G(i,j)=\lim_{n\to\infty}\hat G^{(n)}(i,j)_n$ is a.s. finite.

As $\hat{G}^{(n)}(i,i)$ converges a.s. to $\hat{G}(i,i)$, by dominated convergence, and from (6.4.25), for any $h \geq 0$,

$$\begin{aligned}\mathbb{P}(\hat{G}(i,i) \leq h) &= \mathbb{P}(\lim_{n\to\infty} \hat{G}^{(n)} \leq h)\\ &= \lim_{n\to\infty} \mathbb{P}(\hat{G}^{(n)}(i,i) \leq h)\\ &\geq \lim_{n\to\infty} \mathbb{P}(G^{(n)}(i,i) \leq h)\\ &= \mathbb{P}(\frac{1}{2\gamma} \leq h),\end{aligned}$$

since by Theorem C, $G^{(n)}(i,i) \overset{\text{law}}{=} \frac{1}{2\gamma}$ where γ is a Gamma($\frac{1}{2}$, 1) distributed random variable. Therefore, $\hat{G}(i,i) < \infty$ a.s. For the off diagonal term, as $H_\beta^{(n)}$ is an M-matrix, in particular, $(H_\beta^{(n)})_{|V_n}$ is positive definite, we have

$$\hat{G}^{(n)}(i,j) = \left\langle \delta_i, \hat{G}^{(n)}\delta_j \right\rangle \leq \sqrt{\left\langle \delta_i, \hat{G}^{(n)}\delta_i \right\rangle \left\langle \delta_j, \hat{G}^{(n)}\delta_j \right\rangle} = \sqrt{\hat{G}^{(n)}(i,i)\hat{G}^{(n)}(j,j)}$$

therefore, $\hat{G}(i,j) \leq \sqrt{\hat{G}(i,i)\hat{G}(j,j)}$ and $\hat{G}(i,j)$ is a.s. finite. □

Proof of Theorem 6.2.1 ii). From Proposition 6.5.1, we know that $\psi^{(n)}(k)$ is a positive integrable martingale for all $k \in V$. As a positive martingale, $\psi^{(n)}(k)$ converges a.s. to some non-negative integrable random variable $\psi(k)$.

It remains to show that the convergence does not depend on the choice of the exhausting sequence (V_n). Assume that (Ω_n) is another increasing exhausting sequence, we can similarly construct the martingale $\phi^{(n)}(k)$ associated to Ω_n. As (Ω_n) and (V_n) are exhausting, we can construct a subsequence n_k such that the sequence $V_{n_1}, \Omega_{n_2}, V_{n_3}, \dots$ is increasing and thus the sequence $\psi^{(n_1)}(k), \phi^{(n_2)}(k), \psi^{(n_3)}(k), \dots$ is a martingale for all $k \in V$. This martingale converges a.s. and this identifies the limits of $\psi^{(n)}(k)$ and $\phi^{(n)}(k)$. □

6.6.2 Representation as a mixture of the VRJP on the infinite graph

Firstly, by Proposition (6.4.1) we have

$$G^{(n)}(i,j) = \hat{G}^{(n)}(i,j) + \psi^{(n)}(i)\psi^{(n)}(j)G^{(n)}(\delta_n, \delta_n).$$

From the coupling of Section 6.4.2, and Theorem 6.2.1 i) and ii), we have that a.s.

$$\lim_{n\to\infty} G^{(n)}(i,j) = G(i,j), \tag{6.6.52}$$

where $G(i,j)$ is defined in Theorem 6.2.1 iii).

The next corollary of Proposition 6.3.2 gives the necessary uniform integrability to extend the representation of the VRJP on finite graphs to infinite graphs.

Corollary 6.6.1. *For any $i,j \in V$, there exists $n_0 \in \mathbb{N}$, such that the family of random variable $\frac{G^{(n)}(i_0,j)}{G^{(n)}(i_0,i)}$, $n \geq n_0$ is uniformly integrable.*

Proof. Apply Proposition 6.3.2 by choosing n_0 such that for any $n \geq n_0$, $i,j \in V_n$, denote $K = d_{\mathcal{G}}(i,j)$, we have, for some constant $c > 0$,

$$\mathbb{E}((\frac{G^{(n)}(i_0,j)}{G^{(n)}(i_0,i)})^2) \leq c \cdot \mathbb{E}(\exp(\frac{\eta}{2}(\frac{G^{(n)}(i_0,i)}{G^{(n)}(i_0,j)} + \frac{G^{(n)}(i_0,j)}{G^{(n)}(i_0,i)}))) \leq 2^{K/2}c.$$

The family is uniformly bounded in L^2, in particular uniformly integrable. □

Consider now a connected finite subset $\Lambda \subset V$ containing i_0 and set

$$\partial^+\Lambda = \{j \in \Lambda^c,\ \exists i \in \Lambda \text{ such that } i \sim j\ \}.$$

Let T be the following stopping time

$$T = \inf\{t \geq 0,\ Z_t \notin \Lambda\}.$$

By construction, the distribution of Z_t on $\mathcal{G}$ up to time T equals to the distribution of Z_t on $\mathcal{G}_n$ up to time T, for all n such that $\Lambda \cup \partial^+\Lambda \subset V_n$. We denote by

$$\ell_i(T) = \int_0^T \mathbb{1}_{Z_u=i}\, du,$$

the local time of Z up to time T. Using Theorem C and the coupling of Section 6.4.2, the time-changed VRJP (Z_t) on $\mathcal{G}_n$, starting at i_0, is a mixture of Markov jump process with jumping rates from i to j

$$\frac{1}{2}W^{(n)}_{i,j}\frac{G^{(n)}(i_0,j)}{G^{(n)}(i_0,i)}. \tag{6.6.53}$$

We denote by

$$\tilde{\beta}^{(n)}_i = \sum_{j\sim i}\frac{1}{2}W^{(n)}_{i,j}\frac{G^{(n)}(i_0,j)}{G^{(n)}(i_0,i)},$$

the holding time at site i. We denote by $P^{MJP}_{i_0}$ the law of the Markov Jump process with jump rates $\frac{1}{2}W_{i,j}$ starting from i_0. By simple computation, the Radon-Nykodim derivative of the law of $(Z_t)_{t\leq T}$ under the Markov jump process with jump rates (6.6.53) and under $P^{MJP}_{i_0}$ is

$$e^{-\sum_{i\in\Lambda}\ell_i(T)(\tilde{\beta}^{(n)}_i-\frac{1}{2}W_i)}\frac{G^{(n)}(i_0,Z_T)}{G^{(n)}(i_0,i_0)}$$

where as usual $W_i = \sum_{j\sim i}W_{i,j}$. It implies that for all positive bounded test function F.

$$\begin{aligned}
&\mathbb{E}^{VRJP,\mathcal{G}_n}_{i_0}\left(F((Z_t)_{t\leq T})\right)\\
&= \int\sum_{j\in\partial^+\Lambda}E^{MJP}_{i_0}\left(\mathbb{1}_{Z_T=j}F((Z_t)_{t\leq T})e^{-\sum_{i\in\Lambda}\ell_i(T)(\tilde{\beta}^{(n)}_i-\frac{1}{2}W_i)}\frac{G^{(n)}(i_0,j)}{G^{(n)}(i_0,i_0)}\right)\nu^{\mathcal{G},W}(d\beta,d\gamma)\\
&= \sum_{j\in\partial^+\Lambda}E^{MJP}_{i_0}\left(\mathbb{1}_{Z_T=j}F((Z_t)_{t\leq T})\int e^{-\sum_{i\in\Lambda}\ell_i(T)(\tilde{\beta}^{(n)}_i-\frac{1}{2}W_i)}\frac{G^{(n)}(i_0,j)}{G^{(n)}(i_0,i_0)}\nu^{\mathcal{G},W}(d\beta,d\gamma)\right)
\end{aligned}$$

where $\nu^{\mathcal{G},W}(d\beta,d\gamma)$ is the joint law of β,γ, defined in Theorem 6.2.1 and Section 6.4.2. From (6.6.52), we have a.s.

$$\lim_{n\to\infty}\tilde{\beta}^{(n)}_i = \tilde{\beta}_i = \sum_{j\sim i}\frac{1}{2}W_{i,j}\frac{G(i_0,j)}{G(i_0,i)}$$

Letting n go to infinity, using the uniform integrability of $\frac{G^{(n)}(i_0,j)}{G^{(n)}(i_0,i_0)}$, Corollary 6.6.1, we get that

$$\begin{aligned}
&\mathbb{E}^{VRJP,\mathcal{G}}_{i_0}\left(F((Z_t)_{t\leq T})\right)\\
&= \sum_{j\in\partial^+\Lambda}E^{MJP}_{i_0}\left(\mathbb{1}_{Z_T=j}F((Z_t)_{t\leq T})\int e^{-\sum_{i\in\Lambda}\ell_i(T)(\tilde{\beta}_i-\frac{1}{2}W_i)}\frac{G(i_0,j)}{G(i_0,i_0)}\nu^{\mathcal{G},W}(d\beta,d\gamma)\right)\\
&= \int E^{\beta,\gamma,i_0}_{i_0}\left(F((Z_t)_{t\leq T})\right)\nu^{\mathcal{G},W}(d\beta,d\gamma)
\end{aligned}$$

where $E^{\beta,\gamma,i_0}_{i_0}$ is the expectation associated with the probability $P^{\beta,\gamma,i_0}_{i_0}$ defined in Theorem 6.2.1. This concludes the proof of iii) of that Theorem.

6.6.3 Proof of Proposition 6.2.2, and iv) of Theorem 6.2.1

Proof of Proposition 6.2.2. Recall (6.3.13) and (6.3.14). As $n \mapsto \hat{G}^{(n)}(i,j)$ is increasing, we have

$$\hat{G}(i,j) = \sum_{\sigma \in \mathcal{P}^V_{i,j}} \frac{W_\sigma}{2\beta_\sigma}.$$

By arguments similar to (6.3.17), we have

$$\frac{\hat{G}(i_0,i)}{\hat{G}(i_0,i_0)} = \sum_{\sigma \in \bar{\mathcal{P}}^V_{i,i_0}} \frac{W_\sigma}{(2\beta)^-_\sigma}.$$

Therefore, if we denote $\{(\tilde{Z}_n) \sim \sigma\} = \{\tilde{Z}_0 = \sigma_0, \ldots, \tilde{Z}_m = \sigma_m\}$ with $m = |\sigma|$, then for $i \neq i_0$

$$\begin{aligned} h(i) := P_i^{\beta,\gamma,i_0}(\tau_{i_0} < \infty) &= \sum_{\sigma \in \bar{\mathcal{P}}^V_{i,i_0}} P_i^{\beta,\gamma,i_0}((\tilde{Z}_n) \sim \sigma) \\ &= \sum_{\sigma \in \bar{\mathcal{P}}^V_{i,i_0}} \frac{W_\sigma}{(2\beta)^-_\sigma} \frac{G(i_0,i_0)}{G(i_0,i)} = \frac{\hat{G}(i_0,i)}{\hat{G}(i_0,i_0)} \cdot \frac{G(i_0,i_0)}{G(i_0,i)}. \end{aligned} \tag{6.6.54}$$

It follows from $G(i,j) = \hat{G}(i,j) + \frac{1}{2\gamma}\psi(i)\psi(j)$ that, for $i \neq i_0$,

$$\begin{aligned} P_i^{\beta,\gamma,i_0}(\tau_{i_0} = \infty) &= 1 - h(i) \\ &= \frac{\psi(i_0)}{2\gamma} \frac{\hat{G}(i_0,i_0)\psi(i) - \hat{G}(i_0,i)\psi(i_0)}{\hat{G}(i_0,i_0)G(i_0,i)}. \end{aligned}$$

Therefore,

$$\begin{aligned} P_{i_0}^{\beta,\gamma,i_0}(\tau^+_{i_0} = \infty) &= \sum_{j \sim i_0} \frac{W_{i_0,j} G(i_0,j)}{2\tilde{\beta}_{i_0} G(i_0,i_0)} P_j^{\beta,\gamma,i_0}(\tau_{i_0} = \infty) \\ &= \sum_{j \sim i_0} \frac{\psi(i_0) W_{i_0,j}}{4\gamma \tilde{\beta}_{i_0}} \frac{\hat{G}(i_0,i_0)\psi(j) - \hat{G}(i_0,j)\psi(i_0)}{\hat{G}(i_0,i_0)G(i_0,i_0)}. \end{aligned}$$

Taking the limit $n \to \infty$ in (6.4.22), we have $H_\beta \hat{G}(i_0,\cdot) = \mathbb{1}_{i_0}$. By (iii) of Theorem 6.2.2 we have $H_\beta \psi(\cdot) = 0$, therefore,

$$\sum_{j \sim i_0} W_{i_0,j}[\psi(j)\hat{G}(i_0,i_0) - \psi(i_0)\hat{G}(i_0,j)] = \psi(i_0),$$

hence

$$P_{i_0}^{\beta,\gamma,i_0}(\tau^+_{i_0} = \infty) = \frac{\psi(i_0)^2}{4\gamma \tilde{\beta}_{i_0} \hat{G}(i_0,i_0) G(i_0,i_0)}.$$

□

Remark 6.6.1. *By maximum principle we can check directly that $\hat{G}(i,i)\psi(j) - \hat{G}(i,j)\psi(i)$ is nonnegative. Indeed, let*

$$h_1^{(n)}(j) := \frac{\psi^{(n)}(j)}{\psi^{(n)}(i)} \hat{G}^{(n)}(i,i), \quad h_2^{(n)}(j) := \hat{G}^{(n)}(i,j).$$

We have $h_1^{(n)}(i) = h_2^{(n)}(i)$, $h_1^{(n)}(\delta_n) \geq h_2^{(n)}(\delta_n)$ and $H_\beta^{(n)} h_\cdot^{(n)} = 0$ outside $\{i,\delta_n\}$ for $\cdot \in \{1,2\}$, which means that $h_1^{(n)}, h_2^{(n)}$ are $H_\beta^{(n)}$-harmonic, and $h_1^{(n)} \geq h_2^{(n)}$ on the boundary. This implies that $h_1^{(n)} \geq h_2^{(n)}$, and the inequality by letting n go to ∞.

Proof of Theorem 6.2.1, (iv). From Proposition 6.2.2, we see that $P_{i_0}^{\beta,\gamma,i_0}(\tau_{i_0}^+ = \infty) > 0$ if and only if $\psi(i_0) > 0$. Since the Markov jump process $P_{i_0}^{\beta,\gamma,i_0}$ is irreducible ($\mathcal{G}$ is connected), it implies (iv). □

6.6.4 Ergodicity and the 0-1 law : proof of Proposition 6.2.3 and 6.2.5

Proof of Proposition 6.2.3. From the expression of the Laplace transform of β, c.f. Proposition 6.2.1, we see that β is stationary for the action of $\mathcal{A}$. By 1-dependence, c.f. Proposition 6.2.1, it is also ergodic. Indeed, assume that $(\tau_n) \in \mathcal{A}^{\mathbb{N}}$ is a sequence of automorphims such that $d_{\mathcal{G}}(i_0, \tau_n(i_0)) \to \infty$ for some vertex i_0. We prove that (τ_n) is mixing in the sense that for all $A, B \in \sigma(\beta_i, i \in V)$

$$\lim_{n\to\infty} \mathbb{P}(\tau_n^{-1}(B) \cap A) = \mathbb{P}(A)\mathbb{P}(B),$$

which clearly implies ergodicity. Assume that $V_1 \subset V$ is finite and that $A, B \in \sigma(\beta_j,\ j \in V_1)$. By 1-dependence, $\tau_n^{-1}(B)$ is independent of A for n large enough. Hence, the property is true for $\sigma(\beta_j,\ j \in V_1)$-measurable sets. It can be extended by a monotone class argument.

Since ψ and $\hat{G}$ are constructed as almost sure limit from functions of β, and since the limit does not depend on the choice of the approximating sequence, then ψ and $\hat{G}$ are stationary and ergodic for the action of $\mathcal{A}$.

The event $\{\psi(i) = 0,\ \forall i \in V\}$ is clearly invariant by $\mathcal{A}$, hence has probability 0 or 1. Together with (iv) of Theorem 6.2.1 it concludes the proof of the proposition. □

Proof of Proposition 6.2.5. It works exactly in the same way, using Lemma 6.2.4. □

6.6.5 Proof of Theorem 6.2.2: relation with spectral properties of the random schrödinger operator

Proof of Theorem 6.2.2 (i). As $H_\beta^{(n)} > 0$ a.s., we have that $(H_\beta)_{|V_n\times V_n} > 0$ and passing to the limit, we get $H_\beta \geq 0$. Hence, $\sigma(H_\beta) \subset [0, +\infty)$. □

Proof of Theorem 6.2.2 (ii). As ε is strictly outside the spectrum of $H_\beta + \varepsilon$, the equation $(H_\beta + \varepsilon)\hat{G}_\varepsilon = 1$ has unique finite solution, we can verify by hand that $\sum_{\sigma\in\mathcal{P}_{i,j}^V} \frac{W_\sigma}{(2\beta+\varepsilon)_\sigma}$ is a solution to this equation. Now by Theorem 6.2.1 (i), we have

$$\hat{G}_\varepsilon(i,j) = \sum_{\sigma\in\mathcal{P}_{i,j}^V} \frac{W_\sigma}{(2\beta+\varepsilon)_\sigma} \leq \hat{G}(i,j) < \infty.$$

Therefore, as $\sum_{\sigma\in\mathcal{P}_{i,j}} \frac{W_\sigma}{(2\beta+\varepsilon)_\sigma}$ is increasing as $\varepsilon \to 0$, it converges a.s. to $\hat{G}(i,j)$. Moreover, it can be verified by direct computation on sums of path that $H_\beta\hat{G} = 1$. □

Proof of Theorem 6.2.2 (iii). We have, for all $i \in V_n$,

$$\psi^{(n)}(i) = \sum_{j\sim i} \frac{W_{i,j}}{2\beta_i}\psi^{(n)}(j).$$

As $\psi^{(n)}(i)$ converges a.s. to ψ, the above equality holds in the limit, i.e., for all $i \in V$,

$$\psi(i) = \sum_{j\sim i} \frac{W_{i,j}}{2\beta_i}\psi(j),$$

this exactly means $(H_\beta\psi)(i) = 0$. □

Proof of Theorem 6.2.2 (iv). By Fatou's lemma, the limit $\psi(i)$ satisfies $\mathbb{E}(\psi(i)) \le 1$. By Markov inequality

$$\mathbb{P}(\psi(i) \ge C\|i\|^p) \le \frac{1}{C\|i\|^p}.$$

Let $\partial B(0,n)$ be the sphere of radius n, i.e. $\partial B(0,n) = \{j \in \mathbb{Z}^d,\ d(0,j) = n\}$, when $p > d$.

$$\begin{aligned}\sum_{i\in\partial B(0,n)} \mathbb{P}(\psi(i) \ge C\|i\|^p) &\le \sum_{i\in\partial B(0,n)} \frac{1}{C\|i\|^p}\\ &\le C' \sum_n \frac{n^{d-1}}{n^p} < \infty\end{aligned}$$

for some constant $C' > 0$. By Borel-Cantelli lemma, a.s. only a finite number of i satisfies $\psi(i) \ge C\|i\|^p$. $\square$

6.7 h-transforms

Corollary 6.7.1. *(i) The quenched process (Z_t) on $\mathcal{G}$, conditionally on $\{\tau_{i_0}^+ < \infty\}$, up to its first return time to i_0, is equal in law to the Markov jump process of jump rate from i to j*

$$\begin{cases}\frac{1}{2}W_{i,j}\frac{\hat G(i_0,j)}{\hat G(i_0,i)} & i \ne i_0\\ \tilde\beta_{i_0}\frac{W_{i_0,j}\hat G(i_0,j)}{\sum_{k\sim i_0} W_{i_0,k}\hat G(i_0,k)} & i = i_0,\ j\sim i_0\end{cases}$$

where as usual $\tilde\beta_{i_0} = \sum_{j\sim i_0}\frac{1}{2}W_{i_0,j}\frac{G(i_0,j)}{G(i_0,i_0)}$. Its law is denoted $\hat P_{i_0}^{\beta,i_0}$ in the sequel.

(ii) The annealed process (Z_t) conditionally on $\{\tau_{i_0}^+ < \infty\}$, up to its first return time to i_0, is given by the following mixture :

$$\mathbb{P}_{i_0}^{VRJP}((Z_t)_{t\le\tilde\tau_{i_0}^+} \in \cdot\ |\tau_{i_0}^+ < \infty) = \int \hat P_{i_0}^{\beta,i_0}((Z_t)_{t\le\tilde\tau_{i_0}^+} \in \cdot)\frac{P_{i_0}^{\beta,\gamma,i_0}(\tau_{i_0}^+ < \infty)}{\mathbb{P}_{i_0}^{VRJP}(\tau_{i_0}^+ < \infty)}d\mu_{i_0}^{\mathcal{G},W}(\beta,\gamma),$$

where $\tilde\tau_{i_0}^+ = \inf\{t\ge 0,\ \ Z_t = i_0,\ \exists 0 < s < t \text{ s.t. } Z_s \ne i_0\}$ is the first return time to i_0 of the continuous process (Z_t).

(iii) The quenched process (Z_t), conditionally on the event $\{\tau_{i_0}^+ = \infty\}$, is the Markov jump process with jump rate from i to j

$$\begin{cases}\frac{1}{2}W_{i,j}\frac{\check G(i_0,j)}{\check G(i_0,i)} & i \ne i_0,\ j \ne i_0\\ \tilde\beta_{i_0}\frac{W_{i_0,j}\check G(i_0,j)}{\sum_{k\sim i_0} W_{i_0,k}\check G(i_0,k)} & i = i_0,\ j\sim i_0\\ 0 & i\sim i_0\ j = i_0\end{cases}$$

where $\check G(i,j) = \hat G(i,i)\psi(j) - \hat G(i,j)\psi(i)$. Denote by $\check P_{i_0}^{\beta,\gamma,i_0}$ the law of this Markov process.

(iv) *The annealed process (Z_t) conditionally on the event $\{\tau_{i_0}^+ = \infty\}$, is a mixture of Markov jump process with mixing law*

$$\mathbb{P}_{i_0}^{VRJP}(\,\cdot\,|\tau_{i_0}^+ = \infty) = \int \check{P}_{i_0}^{\beta,\gamma,i_0}(\cdot)\frac{P_{i_0}^{\beta,\gamma,i_0}(\tau_{i_0}^+ = \infty)d\mu_{i_0}^{\mathcal{G},W}(\beta,\gamma)}{\mathbb{P}_{i_0}^{VRJP}(\tau_{i_0}^+ = \infty)}.$$

Remark 6.7.1. *Note that in the case (i), the conditional jump rates do not depend on γ.*

Proof of Corollary 6.7.1. (i) Recall from (6.6.54) that for $i \neq i_0$

$$h(i) = P_i^{\beta,\gamma,i_0}(\tau_{i_0} < \infty) = \frac{\hat{G}(i_0,i)G(i_0,i_0)}{\hat{G}(i_0,i_0)G(i_0,i)}.$$

and $h(i_0) = 1$. For $i \neq i_0$, we have

$$P^{\beta,\gamma,i_0}(\,X_{t+dt} = j\,|X_t = i,\ t \le \tau_{i_0}^+ < \infty) \sim \frac{h(j)}{h(i)}P^{\beta,\gamma,i_0}(\,X_{t+dt} = j\,|X_t = i)$$

Hence, the jumping rate of $P^{\beta,\gamma,i_0}(\,\cdot\,|\tau_{i_0}^+ < \infty)$, up to time $\tau_{i_0}^+$, from i to j is

$$\frac{1}{2}W_{i,j}\frac{G(i_0,j)}{G(i_0,i)}\frac{h(j)}{h(i)} = \frac{1}{2}W_{i,j}\frac{\hat{G}(i_0,j)}{\hat{G}(i_0,i)}.$$

The jumping rate of $P^{\beta,\gamma,i_0}(\,\cdot\,|\tau_{i_0}^+ < \infty)$, up to time $\tau_{i_0}^+$, from i_0 to j is given by

$$\frac{1}{2}W_{i_0,j}\frac{G(i_0,j)}{G(i_0,i_0)}\frac{h(j)}{P_{i_0}^{\beta,\gamma,i_0}(\tau_{i_0}^+ < \infty)} = \tilde{\beta}_{i_0}\frac{W_{i_0,j}\hat{G}(i_0,j)}{\sum_{k\sim i_0}W_{i_0,k}\hat{G}(i_0,k)}.$$

where $\tilde{\beta}_{i_0} = \sum_{l\sim i_0}\frac{1}{2}W_{i_0,l}\frac{G(i_0,l)}{G(i_0,i_0)}$.

(ii) Recall that $\mathbb{P}_{i_0}^{\mathrm{VRJP}}$ denotes the probability of VRJP starting at i_0, and $\nu^{\mathcal{G},W}$ the joint law of (β,γ).

$$\begin{aligned}
&\mathbb{P}_{i_0}^{\mathrm{VRJP}}(\,(Z_t)_{t\le\tau_{i_0}^+} \in \cdot\,|\tau_{i_0}^+ < \infty)\\
&= \int P_{i_0}^{\beta,\gamma,i_0}(\,(Z_t)_{t\le\tau_{i_0}^+} \in \cdot\,|\tau_{i_0}^+ < \infty)\frac{P_{i_0}^{\beta,\gamma,i_0}(\tau_{i_0}^+ < \infty)}{\mathbb{P}_{i_0}^{\mathrm{VRJP}}(\tau_{i_0}^+ < \infty)}\nu^{\mathcal{G},W}(d\beta,d\gamma)\\
&= \int \hat{P}_{i_0}^{\beta,i_0}((Z_t)_{t\le\tau_{i_0}^+} \in \cdot)\frac{P_{i_0}^{\beta,\gamma,i_0}(\tau_{i_0}^+ < \infty)}{\mathbb{P}_{i_0}^{VRJP}(\tau_{i_0}^+ < \infty)}\nu^{\mathcal{G},W}(d\beta,d\gamma).
\end{aligned}$$

(iii) Similarly to (i), for $i \neq i_0$, we have

$$P^{\beta,\gamma,i_0}(\,X_{t+dt} = j\,|X_t = i,\ \tau_{i_0}^+ = \infty) \sim \frac{1-h(j)}{1-h(i)}P^{\beta,\gamma,i_0}(\,X_{t+dt} = j\,|X_t = i)$$

Hence, the jumping rate of $P^{\beta,\gamma,i_0}(\,\cdot\,|\tau_{i_0}^+ = \infty)$, from i to j is

$$\frac{1}{2}W_{i,j}\frac{G(i_0,j)}{G(i_0,i)}\frac{1-h(j)}{1-h(i)} = \frac{1}{2}W_{i,j}\frac{\hat{G}(i_0,i_0)\psi(j) - \hat{G}(i_0,j)\psi(i_0)}{\hat{G}(i_0,i_0)\psi(i) - \hat{G}(i_0,i)\psi(i_0)} = \frac{1}{2}W_{i,j}\frac{\check{G}(i_0,j)}{\check{G}(i_0,i)}.$$

The jumping rate of $P^{\beta,\gamma,i_0}(\,\cdot\,|\tau_{i_0}^+ = \infty)$, from i_0 to j is given by

$$\frac{1}{2}W_{i_0,j}\frac{G(i_0,j)}{G(i_0,i_0)}\frac{1-h(j)}{P^{\beta,\gamma,i_0}(\tau_{i_0}^+=\infty)} = \tilde{\beta}_{i_0}\frac{W_{i_0,j}\check{G}(i_0,j)}{\sum_{k\sim i_0} W_{i_0,k}\check{G}(i_0,k)}.$$

where $\tilde{\beta}_{i_0} = \sum_{l\sim i_0}\frac{1}{2}W_{i_0,l}\frac{G(i_0,l)}{G(i_0,i_0)}$.
(iv) follows easily from (iii) in the same way as in (ii).

□

6.8 Proof of recurrence of 2-dimensional ERRW : Theorem 6.2.5

Consider the cubical graph $\mathcal{G} = (\mathbb{Z}^2, E)$ with constant edge weight $a_e = a > 0$. We apply the abstract lemma 2.5 of [37]. Let $\ell \in \mathbb{Z}^2$. In order to apply lemma 2.5 to $v_0 = 0$ and $v_1 = \ell$, we need to have a transformation that leaves invariant the graph and its weights and that exchanges v_0 and v_1. Take $V_n = B(\frac{\ell}{2}, n)$, the ball with center $\ell/2$ and radius n. Consider as in Section 6.4.2 the graph

$$\mathcal{G}_n = (\tilde{V}_n = V_n \cup \{\delta_n\}, E_n),$$

and the associated weights $(a_e^{(n)})_{e\in E_n}$ obtained by restriction to V_n with wired boundary condition. Clearly, central symmetry[2] with respect to $\frac{\ell}{2}$ leaves invariant $(\mathcal{G}^{(n)}, a^{(n)})$ and exchange 0 and ℓ.

Following the discussion of Section 6.2.4, we consider for $i \sim j$ in $\mathbb{Z}^2$

$$x_{i,j} = W_{i,j}G(0,i)G(0,j),$$

where (W,β,γ) are distributed according to $\tilde{\nu}^{\mathcal{G},a}(dW, d\beta, d\gamma)$. With the coupling defined in Section 6.4.2, we define for $i \sim j$, i, j in $\tilde{V}_n$,

$$x_{i,j}^{(n)} = W_{i,j}^{(n)}G^{(n)}(0,i)G^{(n)}(0,j).$$

where $W^{(n)}$ is obtained by restriction with wired boundary condition from W. By additivity of Gamma random variables, $(W_e^{(n)})_{e\in E_n}$ are independent Gamma random variables with parameters $(a_e^{(n)})_{e\in E_n}$. Hence, the ERRW on $\tilde{V}_n$, with initial weights $a^{(n)}$, starting from 0, is a mixture of reversible Markov chains with conductances $(x_e^{(n)})_{e\in E_n}$.

Moreover, from Theorem 6.2.1, we have that for all $i, j \in \mathbb{Z}^2$, $i \sim j$, a.s.

$$\lim_{n\to\infty} x_{i,j}^{(n)} = x_{i,j}. \tag{6.8.55}$$

Let $\phi : E_n \mapsto [0,1]$, be a function such that $\phi(e) = 0$ if $0 \in e$ and $\phi(e) = 1$ if $\ell \in e$. Then lemma 2.5 of [37] asserts that

$$\mathbb{E}\left(\left(\frac{x_\ell^{(n)}}{x_0^{(n)}}\right)^{\frac{1}{4}}\right) \le \exp\left(-\frac{1}{32S_\phi}\right),$$

where as usual for $i \in \tilde{V}_n$, $x_i = \sum_{j\sim\ell} x_{\ell,j}^{(n)}$, and where

$$S_\phi = \sum_{i\in\tilde{V}_n}\frac{a_i+1}{2}\max_{\substack{e,e'\in E_n,\\ i\in e, i\in e'}}(\phi(e)-\phi(e'))^2.$$

[2]which leaves δ_n invariant.

In the sequel we choose ϕ compactly supported and consider n large, thus we can consider S_ϕ does not depend on n.

Consider now $B_\ell = B(\ell, \frac{1}{2}|\ell|_\infty)$ the $|\cdot|_\infty$ ball with center ℓ and radius $\frac{1}{2}|\ell|_\infty$. Take n large enough so that V_n^c is at distance at least 2 from B_ℓ. Consider

$$h(i) = P_i^{SRW}\left(H_\ell < H_{B_\ell^c}\right),$$

where P^{SRW} is the law of the simple random walk on $\mathbb{Z}^2$ and H_ℓ, resp. $H_{B_\ell^c}$, is the first hitting time of ℓ, resp. B_ℓ^c. Clearly $h(\ell) = 1$, $h(0) = 0$, and h is harmonic in $B_\ell \setminus \{\ell\}$. Moreover, it is classical (cf [33] section 2.1) that

$$\mathcal{E}(h,h) := \frac{1}{2}\sum_{i\sim j}(h(i) - h(j))^2 = \frac{1}{R(\ell, B_\ell^c)},$$

where $R(\ell, B_\ell^c)$ is the equivalent resistance between ℓ and B_ℓ^c. Moreover in dimension $d = 2$ (cf Proposition 2.14 of [33]), we have

$$R(\ell, B_\ell^c) \asymp C \log\left(\frac{|\ell|_\infty}{2}\right),$$

when $|\ell| \to \infty$, for some constant $C > 0$.

Consider now the function ϕ such that

$$\phi(e) = \begin{cases} 0, & \text{if } 0 \in e \\ 1, & \text{if } \ell \in e \\ \frac{1}{2}(h(i) + h(j)), & \text{otherwise, with } e = \{i,j\} \end{cases}$$

The following lemma is rather elementary.

Lemma 6.8.1. *In the case of constant weights $a_e = a$, for n large enough*

$$S_\phi \le 10(a+1)\mathcal{E}(h,h).$$

This implies that

$$\mathbb{E}\left(\left(\frac{x_\ell^{(n)}}{x_0^{(n)}}\right)^{\frac{1}{4}}\right) \le |\ell|_\infty^{-\xi}.$$

with $\xi = \frac{C}{10a}$. By (6.8.55) and Fatou's lemma,

$$\mathbb{E}\left(\left(\frac{x_\ell}{x_0}\right)^{\frac{1}{4}}\right) \le |\ell|_\infty^{-\xi}. \tag{6.8.56}$$

But

$$x_\ell = \sum_{j\sim\ell} W_{\ell,j} G(0,\ell)G(0,j) = 2\beta_\ell G(0,\ell)^2 \ge \frac{\beta_\ell}{2\gamma^2}\psi(0)^2\psi(\ell)^2.$$

Similarly,

$$x_0 = \sum_{j\sim 0} W_{0,j} G(0,0)G(0,j) = G(0,0)(2\beta_0 G(0,0) - 1).$$

Hence,

$$\frac{x_\ell}{x_0} \ge \frac{\psi(0)^2}{2\gamma^2 G(0,0)(2\beta_0 G(0,0) - 1)}\beta_\ell \psi_\ell^2.$$

Assume the ERRW is transient. It implies that, a.s., $\psi(i) > 0$ for all i. Moreover $\beta_\ell \psi_\ell^2$ is stationary with respect to translations. It is incompatible with polynomial decrease (6.8.56). More precisely, since

$$\begin{aligned}
\mathbb{P}(\frac{x_\ell}{x_0} > \epsilon) &\geq \mathbb{P}(\frac{\psi(0)^2}{2\gamma^2 G(0,0)(2\beta_0 G(0,0)-1)}\beta_\ell\psi_\ell^2 > \epsilon) \\
&\geq \mathbb{P}(\frac{\psi(0)^2}{2\gamma^2 G(0,0)(2\beta_0 G(0,0)-1)} > \sqrt{\epsilon}; \beta_\ell\psi_\ell^2 > \sqrt{\epsilon}) \\
&\geq 1 - \mathbb{P}(\frac{\psi(0)^2}{2\gamma^2 G(0,0)(2\beta_0 G(0,0)-1)} \leq \sqrt{\epsilon}) - \mathbb{P}(\beta_\ell\psi_\ell^2 \leq \sqrt{\epsilon}).
\end{aligned}$$

But we are able to find ϵ small enough such that for all ℓ

$$\mathbb{P}(\frac{\psi(0)^2}{2\gamma^2 G(0,0)(2\beta_0 G(0,0)-1)} \leq \sqrt{\epsilon}) \leq 1/4 \text{ and } \mathbb{P}(\beta_\ell\psi_\ell^2 \leq \sqrt{\epsilon}) \leq 1/4.$$

6.9 Proof of Functional central limit theorems for the VRJP and the ERRW : Theorem 6.2.3 and 6.2.4

Proof of Theorem 6.2.3 and Theorem 6.2.4. Let us start by the VRJP with constant weights $W_{i,j} = W$. Assume that the VRJP is transient.

Denote by $(X_n)_{n\in\mathbb{N}}$ the canonical process on $(\mathbb{Z}^d)^{\mathbb{N}}$. Given the environment β,γ, let us define $\tilde{P}^\psi$ to be the law of the reversible Markov chain with conductances $W_{i,j}\psi(i)\psi(j)$, i.e. with transition probabilities

$$\tilde{P}^\psi(X_{n+1} = j | X_n = i) = \frac{W_{i,j}\psi(j)}{\sum_{l\sim i} W_{i,l}\psi(l)}.$$

Denote by $\tilde{P}^{\beta,\gamma,0}$ the law of the underlying discrete time process associated with the Markov Jump process $P^{\beta,\gamma,0}$, so that for $i \sim j$

$$\tilde{P}^{\beta,\gamma,0}(X_{n+1} = j | X_n = i) = \frac{W_{i,j}G(0,j)}{\sum_{l\sim i} W_{i,l}G(0,l)}.$$

As ψ is a generalized eigenfunction of H_β, for any $i \in V$

$$\beta_i = \sum_{j\sim i} \frac{1}{2} W_{i,j}\frac{\psi(j)}{\psi(i)}.$$

It then follows that, for $i \neq 0$

$$\begin{aligned}
h^\psi(i) := \tilde{P}_i^\psi(\tau_0 < \infty) &= \sum_{\sigma\in\bar{\mathcal{P}}_{i,0}^V} \tilde{P}_i^\psi(Z_n \sim \sigma) \\
&= \sum_{\sigma\in\bar{\mathcal{P}}_{i,0}^V} \frac{W_\sigma}{(2\beta)_\sigma^-}\frac{\psi(0)}{\psi(i)} = \frac{\hat{G}(0,i)}{\hat{G}(0,0)}\frac{\psi(0)}{\psi(i)}.
\end{aligned}$$

Consider the Markov chain $\tilde{P}_0^\psi(\,\cdot\,|\tau_0^+ = \infty)$ (Doob's $(1-h^\psi)$-transform). By similar computation as in the proof of Proposition 6.7.1, we have that the transition probability of $\tilde{P}_0^\psi(\,\cdot\,|\tau_0^+ = \infty)$ from i to j is, for $j \neq 0$,

$$\frac{W_{i,j}\psi(j)(1-h^\psi(j))}{\sum_{l\sim i} W_{i,l}\psi(l)(1-h^\psi(l))} = \frac{W_{i,j}\check{G}(0,j)}{\sum_{l\sim i} W_{i,l}\check{G}(0,l)}$$

and 0 when $j = 0$. Therefore, we see that the transition probabilities of $\tilde{P}_0^{\psi}(\,\cdot\,|\tau_0^+ = \infty)$ are the same as those of $\tilde{P}_0^{\beta,\gamma,0}(\,\cdot\,\mid \tau_0^+ = \infty)$, cf iii) of Proposition 6.7.1. Moreover, if we denote

$$\xi_0 = \sup\{n;\ X_n = 0\}$$

then by strong Markov property

$$\tilde{P}_0^{\psi}(X_n \in \cdot|\tau_0^+ = \infty) = \tilde{P}_0^{\psi}((Y \circ \theta_{\xi_0})_n \in \cdot)$$

$$\tilde{P}_0^{\beta,\gamma,0}(X_n \in \cdot|\tau_0^+ = \infty) = \tilde{P}_0^{\beta,\gamma,0}((X \circ \theta_{\xi_0})_n \in \cdot)$$

where θ_n is the shift in time by n. It follows that $(X \circ \theta_{\xi_0})_n$ has the same law under $\tilde{P}_0^{\psi}$ and under $\tilde{P}_0^{\beta,\gamma,0}$.

Remark also, from Proposition 6.2.3, that $W_{i,j}\psi(i)\psi(j)$. are stationary and ergodic conductances. We can thus apply Theorem 4.5 and Theorem 4.6 of [34]. In order to have a functional central limit theorem we need to show that, cf Theorem 4.5 of [34],

$$\mathbb{E}(W_{i,j}\psi(i)\psi(j)) < \infty. \tag{6.9.57}$$

In order to show that it has non-degenerate asymptotic covariance we need to show that, cf Theorem 4.6 and identity (4.20) of [34],

$$\mathbb{E}\left(\frac{1}{W_{i,j}\psi(i)\psi(j)}\right) < \infty. \tag{6.9.58}$$

By invariance of the law of the conductances by symmetries of $\mathbb{Z}^d$, we know that the limit diffusion matrix is of the form $\sigma^2\mathrm{Id}$.

The same reasoning works in the case of the ERRW with constant weights $a_{i,j} = a$: in this case $(W_{i,j})$ are i.i.d., but as shown in Proposition 6.2.5, $W_{i,j}\psi(i)\psi(j)$ is also stationary and ergodic under $\tilde{\nu}^a(dW, d\beta)$.

Estimates (6.9.57) and (6.9.58) are provided by [23] in the VRJP case, and by [22] in the ERRW case. This is summarized in the following lemma.

Lemma 6.9.1. *(i) (VRJP case) Consider the VRJP on $\mathbb{Z}^d$, for $d \geq 3$, with constant weights $W_{ij} = W$ There exists $0 < \lambda_2 < \infty$ such that for $W > \lambda_2$, the VRJP is transient and such that (6.9.57), (6.9.58) are true under $\nu^W(d\beta)$.*

(ii) (ERRW case) Consider the ERRW on $\mathbb{Z}^d$, for $d \geq 3$, with constant weights $a_{ij} = a$ There exists $0 < \tilde{\lambda}_2 < \infty$ such that for $a > \tilde{\lambda}_2$, the ERRW is transient and (6.9.57), (6.9.58) are true under $\tilde{\nu}^a(dW, d\beta)$.

The proof of that lemma is given below.

Consider the VRJP case and assume that the condition of the lemma is satisfied. Define

$$X_t^{(n)} = \frac{X_{\lfloor nt \rfloor}}{\sqrt{n}}.$$

From [34], we know that there exists a $0 < \sigma^2 < \infty$ such that for all bounded Liptchitz function F for the Skorokhod topology, for all $\epsilon > 0$, for all $0 < T < \infty$,

$$\lim_{n\to\infty} \mathbb{P}_{Q^*}\left(\left|E_0^{\psi}(F((X^{(n)}_{0\le t\le T})) - \mathbb{E}(F((B_{0\le t\le T}))\right| \geq \epsilon\right) = 0. \tag{6.9.59}$$

where B_t is a d-dimensional Brownian motion with covariance $\sigma^2\mathrm{Id}$, and where Q^* is the invariant measure for the processes viewed from the particle

$$Q^* = \frac{\sum_{j\sim 0} W_{0,j}\psi(0)\psi(j)}{\mathbb{E}_{\nu^W}(\sum_{j\sim 0} W_{0,j}\psi(0)\psi(j))} \cdot \nu^W(d\beta).$$

It is clear, since Q^* and ν^W are equivalent probability distribution that (6.9.59) is also true when $\mathbb{P}_{Q^*}$ is replaced by $\mathbb{P}_{\nu^W}$. This implies an annealed functional central limit theorem for the process (X_n) under the annealed law $\mathbb{E}_{\nu^W}\left(\tilde{P}_0^{\psi}(\cdot)\right)$:

$$\lim_{n\to\infty}\left|\mathbb{E}_{\nu^W}\left(E_0^{\psi}(F((X^{(n)}_{0\le t\le T}))\right)-\mathbb{E}\left(F((B_{0\le t\le T}))\right|=0. \tag{6.9.60}$$

Let $\Upsilon^{(n)}_t := \frac{1}{\sqrt{n}}(X\circ\theta_{\xi_0})_{[nt]}$. Denote d° the Skorohod metric on $D([0,\infty),\mathbb{R}^d)$, the space of càdlag functions $f:[0,\infty)\to\mathbb{R}^d$. As

$$|X^{(n)}_t-\Upsilon^{(n)}_t|=\frac{1}{\sqrt{n}}|X_{[nt]}-X_{[nt+\xi_0]}|\le\frac{|\xi_0|}{\sqrt{n}}\xrightarrow[n\to\infty]{}0,$$

we have

$$d^\circ(X^{(n)},\Upsilon^{(n)})\to 0. \tag{6.9.61}$$

Recall that F is bounded Lipschitz function for the Skorohod topology, therefore,

$$|F(X^{(n)}_t)-F(\Upsilon^{(n)}_t)|\to 0$$

and (6.9.60) is valid for $X^{(n)}$ replaced by $\Upsilon^{(n)}$. But $\Upsilon^{(n)}$ has the same law under $\tilde{P}_0^{\psi}$ and $\tilde{P}_0^{\beta,\gamma,0}$. This implies the functional central limit theorem (6.9.60), for the annealed law $\mathbb{E}_{\nu^W}\left(\tilde{P}_0^{\beta,\gamma,0}(\cdot)\right)$ in place of $\mathbb{E}_{\nu^W}\left(\tilde{P}_0^{\psi}(\cdot)\right)$ starting from 0. But, by Theorem 6.2.1, the annealed law $\mathbb{E}_{\nu^W}\left(\tilde{P}_0^{\beta,\gamma,0}(\cdot)\right)$ is that of the discrete time VRJP.

The proof is exactly the same for the ERRW, one just need to replace the law $\nu^W(d\beta)$ by the law $\tilde{\nu}^a(dW,d\beta)$. □

Proof of Lemma 6.9.1. Let us start by the ERRW case, ii). Consider the sequence of subsets of $\mathbb{Z}^d$, $V_n=[-n,n]^d$. Recall that

$$\psi^{(n)}(j)=e^{u^{(n)}(\delta_n,j)},$$

when $j\in V_n$. Consider the point $y_n=(-n,0,\dots,0)$, so that y_n is at the boundary of the set, $y_n\sim\delta_n$. By Lemma 7 of [22] (which is the ERRW's counterpart of Proposition 6.3.2, Section 6.3.2), we have for $a>16$,

$$\mathbb{E}_{\tilde{\nu}}\left((\cosh(u(\delta_n,y_n))^8\right)\le 2, \tag{6.9.62}$$

(Indeed, the proof does not depend on the graph structure, nor on the choice of the rooting).

From, [22], Theorem 4, there exists $0<\tilde{\lambda}_2<\infty$ such that if $a>\tilde{\lambda}_2$, then for all i,j in V_n,

$$\mathbb{E}_{\tilde{\nu}}\left(\left(\cosh(u^{(n)}(\delta_n,i)-u^{(n)}(\delta_n,j))\right)^8\right)\le 2. \tag{6.9.63}$$

Remark that in [22], the rooting of the field is at 0 and the graph is the restriction of the graph $\mathbb{Z}^d$ to V_n. But an attentive reading of the proof shows that the result is also valid for the graph $\mathcal{G}_n=(V_n\cup\{\delta_n\},E_n)$ and rooting δ_n as well. Indeed, the estimate is based on the protected Ward's estimates, Lemma 4, which remain valid for diamonds inside the set V_n, and on the estimate on effective conductances, Proposition 3, which is in fact an estimate inside a "diamond". Remark that the estimate (6.9.63) is also valid when i or j is at the boundary of the set V_n (in fact the proof is written in the case where the diamond $R_{i,j}$ is inside the set V_n, which is the case when $j = y_n$ and $i \in \mathbb{Z}^d$ fixed for n large enough). Specified to $j = y_n$ and $i \in \mathbb{Z}^d$ fixed, it gives for n large enough

$$\mathbb{E}_{\tilde{\nu}}\left(\left(\cosh(u(\delta_n, i) - u^{(n)}(\delta_n, y_n))\right)^8\right) \leq 2. \tag{6.9.64}$$

By Cauchy-Schwartz inequality, and by (6.9.62) and (6.9.64), we get that

$$\mathbb{E}_{\tilde{\nu}^a}\left((\psi^{(n)}(i))^{\pm 4}\right) \leq \mathbb{E}_{\tilde{\nu}^a}\left(e^{\pm 8u^{(n)}(\delta_n, y_n)}\right)^{\frac{1}{2}} \mathbb{E}_{\tilde{\nu}^a}\left(e^{\pm 8(u^{(n)}(\delta_n, i) - u^{(n)}(\delta_n, y_n))}\right)^{\frac{1}{2}} \leq C_\pm$$

for some constant $C_\pm > 0$ independent of n. From this we deduce by Fatou's lemma for all i, j in $\mathbb{Z}^d$, $i \sim j$,

$$\mathbb{E}_{\tilde{\nu}^a}\left(\left((W_{i,j}\psi(i)\psi(j)\right)^{\pm 1}\right) \leq \mathbb{E}_{\tilde{\nu}^a}\left(\left(W_{i,j}\right)^{\pm 2}\right)^{\frac{1}{2}} \mathbb{E}_{\tilde{\nu}^a}\left((\psi(0))^{\pm 4}\right)^{\frac{1}{2}} < \infty,$$

for a large enough.

The proof is very similar in the VRJP case, and uses Theorem 1 of [23]. As in the proof of the estimate for the ERRW, the estimate is valid in the case we are interested in, that is for the graph $\mathcal{G}_n$, rooted at δ_n, and for $x \in \mathbb{Z}^d$, $y = y_n$ for n large enough.

□

6.10 Proof of technical Lemma 6.5.2

Proof of Lemma 6.5.2. The idea of "dérivée particulaire" (cf e.g. [50]) comes from the theory of fluid dynamics, where one often needs to integrate certain function on time dependent domains. Suppose we are about to integrate on the domain Ω_t of some material, and $\Phi_t(\Omega_0) = \Omega_t$ encodes the motion of the fluid, than it provides a natural change of variables, which enable us to fall back to a constant domain of integration: Ω_0. Here the situation is similar, we define a change of variable that sends the domains Ω_t to a fixed domain $(\mathbb{R}_+)^{N+1}$.

We first prove the particular case (6.5.42) of the identity, since it is slightly simpler to write and contains the main ingredients of the general formula. Recall that $\mathcal{G}_n = (\tilde{V}_n = V_n \cup \{\delta\}, E_n)$ is the induced graph with wired boundary condition.

We first describe the change of variables that was used in the proof of Theorem 1 of [43], and that sends the domain $\Omega = \{\beta_{\tilde{V}_n},\ H_\beta^{(n)} > 0\}$ to the domain $(\mathbb{R}_+ \setminus \{0\})^{\tilde{V}_n}$. We first identify V_n with $\{1, 2, \ldots, |V_n| = N\}$ and δ_n with $N+1$ then

$$H_\beta^{(n)} = \begin{pmatrix} 2\beta_1 & -W_{1,2} & \cdots & -W_{1,\delta_n} \\ -W_{1,2} & 2\beta_2 & \cdots & -W_{2,\delta_n} \\ \cdots & \cdots & \cdots & \cdots \\ -W_{1,\delta_n} & -W_{2,\delta_n} & \cdots & -2\beta_{\delta_n} \end{pmatrix}$$

Define

$$\Phi : \begin{cases} \Omega \to \mathbb{R}^{N+1} \\ \beta_{\tilde{V}_n} \mapsto x_{\tilde{V}_n} \end{cases}$$

by

$$x_1 = 2\beta_1, x_2 = \frac{M_2}{M_1}, \ldots, x_N = \frac{M_N}{M_{N-1}}, x_{\delta_n} = x_{N+1} = \frac{\det H_\beta^{(n)}}{M_N} \tag{6.10.65}$$

where M_k is the k-th leading principal minor of $H_\beta^{(n)}$. In [43] Lemma 1, it is shown that Φ is a diffeomorphism and the reciprocal of Φ is computed explicitly : more precisely, define $\{H_{i,j}, 1 \le i < j \le N+1\}$ recursively by

$$\begin{cases} H_{1,j} = W_{1,j} & j > 1 \\ H_{i,j} = W_{i,j} + \sum_{k=1}^{i-1} \frac{H_{k,i} H_{k,j}}{x_k} & i \ge 2, j > i \end{cases}$$

we have

$$\beta_i = \frac{x_i}{2} + \sum_{k=1}^{i-1} \frac{(H_{k,i})^2}{x_k}.$$

Hence, we can write

$$\begin{aligned} \beta_1 &= \frac{x_1}{2} \\ \beta_2 &= \frac{x_2}{2} + \frac{W_{1,2}^2}{2x_1} + C_2(W_{i,j}, i,j \le 2;\ x_1) \\ \beta_3 &= \frac{x_3}{2} + \sum_{k=1}^{2} \frac{W_{k,3}^2}{2x_k} + C_3(W_{i,j}, i,j \le 3;\ x_1, x_2) \\ &\dots \\ \beta_N &= \frac{x_N}{2} + \sum_{k=1}^{N-1} \frac{W_{k,N}^2}{2x_k} + C_N(W_{i,j}, i,j \le N;\ x_1, \dots, x_{N-1}) \\ \beta_{N+1} = \beta_{\delta_n} &= \frac{x_{N+1}}{2} + \sum_{k=1}^{N} \frac{W_{k,N+1}^2}{2x_k} + C_{N+1}(W_{i,j}, i,j \le N+1;\ x_1, \dots, x_N) \end{aligned} \tag{6.10.66}$$

where C_l is a positive function of $W_{i,j}, i,j \le l;\ x_1, \dots, x_{l-1}$ for $l \in \{2, \dots, N+1\}$. More precisely, C_l can be written as a sum with positive coefficients of terms of the form

$$\frac{W_{e_1} \cdots W_{e_{k'}}}{x_{i_1} \cdots x_{i_{k'}}}, \tag{6.10.67}$$

where $e_1, \dots, e_{e_k}$ are some edges and $i_1, \dots, i_{k'}$ are some vertices in $\{1, \dots, l-1\}$. Note that by inverting $H_\beta^{(n)}$ using cofactors, we have $G(\delta_n, \delta_n) = \frac{M_n}{\det H_\beta^{(n)}} = 1/x_\delta$. Note also that the Jacobian matrix of Φ^{-1} is lower triangular with $\frac{1}{2}$ in the diagonal, hence the Jacobian determinant is $\det(\nabla_x \Phi^{-1}) = \frac{1}{2^{N+1}}$, which does not depend $(W_{i,j})$.

Fix some i_0, j_0, denote $W^t = W + t\mathbb{1}_{i_0,j_0}$, that is, $W^t_{i_0,j_0} = W_{i_0,j_0} + t$ and $W^t_{i,j} = W_{i,j}, \{i,j\} \neq \{i_0, j_0\}$; we choose $t \ge 0$ and we will differentiate w.r.t. t at $t = 0^+$. With such notation we clearly have $\partial_{W_{i_0,j_0}} = \partial_t$. We write $H_{\beta,t}^{(n)}$, the Schrödinger operator associated with potential β and weights W^t. Denote $\Omega_t = \{\beta_{\tilde{V}_n}; H_{\beta,t}^{(n)} > 0\}$. Denote

$$f^{(n)}(\beta_{\tilde{V}_n}, t) = \left(\frac{2}{\pi}\right)^{|\tilde{V}_n|/2} \frac{\exp(-\sum_{i \in \tilde{V}_n} \beta_i + \sum_{E_n} W^t_{i,j} - \Psi(G^{(n)}(\delta_n, \delta_n)))}{\sqrt{\det H_{\beta,t}^{(n)}}}$$

the function defined in (6.5.36) for potential β and weights W^t, which is now considered as a function of $\beta_{\tilde{V}_n}$ and t. We need to show the following:

$$\frac{d}{dt}\int_{\Omega_t} f^{(n)}(\beta_{\tilde{V}_n}, t)d\beta_{\tilde{V}_n} = \int_{\Omega_t} \frac{\partial f^{(n)}}{\partial t}(\beta_{\tilde{V}_n}, t)d\beta_{\tilde{V}_n}. \tag{6.10.68}$$

We denote by Φ_t the change of variables defined previously, for the weights W^t:

$$\Phi_t : \begin{cases} \Omega_t \to \mathbb{R}^{N+1} \\ \beta_{\tilde{V}_n} \mapsto x_{\tilde{V}_n} \end{cases}$$

which enables us to consider $\beta_{\tilde{V}_n}$ (and its derivations) as functions of $x_{\tilde{V}_n}$ and t and vice versa, i.e. we can write e.g.

$$\beta_{\tilde{V}_n}(x_{\tilde{V}_n}, t) = \Phi_t^{-1}(x_{\tilde{V}_n}) \text{ and } x_{\tilde{V}_n}(\beta_{\tilde{V}_n}, t) = \Phi_t(\beta_{\tilde{V}_n}).$$

Let $g : \mathbb{R}^{N+1} \times \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$g(x_{\tilde{V}_n}, t) = f^{(n)}(\Phi_t^{-1}(x_{\tilde{V}_n}), t).$$

As the Jacobian of Φ_t is $\nabla_x \Phi_t^{-1} = \frac{1}{2^{N+1}}$, which does not depend on t, we have, writing $\prod_{i\in\tilde{V}_n} dx_i = dx_{\tilde{V}_n}$ for short

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} f^{(n)}(\beta_{\tilde{V}_n}, t)d\beta_{\tilde{V}_n} &= \frac{d}{dt}\int_{\mathbb{R}^{N+1}} g(x_{\tilde{V}_n}, t)\det(\nabla_x \Phi_t^{-1})dx_{\tilde{V}_n} \\ &= \int_{\mathbb{R}^{N+1}} \frac{\partial}{\partial t}[g(x_{\tilde{V}_n}, t)]\det(\nabla_x \Phi_t^{-1})dx_{\tilde{V}_n}.\end{aligned}$$

To justify the inversion of the derivation and integration, remark that, using (6.10.66),

$$g(x_{\tilde{V}_n}, t) = \frac{(\frac{2}{\pi})^{|\tilde{V}_n|/2}}{\sqrt{x_1 x_2 \cdots x_{N+1}}} \cdot \exp(-\sum_{k=1}^{N+1} \frac{x_k}{2} - \Psi(1/x_{\delta_n}) + \sum_{i,j} W_{i,j}^t - C)$$

with $C = \sum_{1\le i<j\le N+1} \frac{(W_{i,j}^t)^2}{2x_i} + \sum_{l=2}^{N+1} C_l$, where C_l is the function of $(x_k)_{k=1,\dots,l-1}$ and W^t defined in (6.10.66). Note that in the above expression, as the graph is connected, we can choose an arrangement of the vertices in such a way that (for details, c.f. justification of (9) in [43]), for every variable $x_i, i \in V_n$, we have an exponential term $\exp(-\frac{x_i}{2} - \frac{A_i}{2x_i} - \tilde{C}_i)$, where $A_i > 0$ is a positive real number and $\tilde{C}_i$ is some positive function. For the variable x_{δ_n}, as we have added $\Psi(G^{(n)}(\delta_n, \delta_n)) = \Psi(1/x_{\delta_n})$, it is also of the same form since $\Psi(x) = x/2 - 1$.

To recapitulate, let c be some constant, we have

$$g(x_{\tilde{V}_n}, t) \le \frac{c}{\sqrt{x_1 x_2 \cdots x_{N+1}}} \exp(-\sum_{k=1}^{N+1} (\frac{x_i}{2} + \frac{A_i}{2x_i}) + \sum_{i,j} W_{i,j}^t). \tag{6.10.69}$$

Finally, we see that the derivation $\frac{\partial}{\partial t} g(x_{\tilde{V}_n}, t)$ is of the form

$$P(x, W^t) g(x_{\tilde{V}_n}, t),$$

where $P(x, W^t)$ is a finite sum of terms of the form (6.10.67). Hence, $\frac{\partial}{\partial t} g(x_{\tilde{V}_n}, t)$ is dominated by an integrable function, and we can legitimately invert derivation and integral.

On the other hand,

$$\int_{\Omega_t} \frac{d}{dt} f^{(n)}(\beta_{\tilde V_n}, t) d\beta_{\tilde V_n} = \int_{\Omega_t} \frac{d}{dt} g(x_{\tilde V_n}(\beta_{\tilde V_n}, t), t) d\beta_{\tilde V_n}$$

$$= \int_{\Omega_t} \left[\sum_{k=1}^{N+1} \frac{\partial g(x_{\tilde V_n}, t)}{\partial x_k} \frac{\partial x_k(\beta_{\tilde V_n}, t)}{\partial t} + \frac{\partial g(x_{\tilde V_n}, t)}{\partial t} \right] d\beta_{\tilde V_n}$$

where we of course consider, for any $1 \le k \le N+1$,

$$\frac{\partial g(x_{\tilde V_n}, t)}{\partial x_k} = (\frac{\partial g}{\partial x_k})(x_{\tilde V_n}(\beta_{\tilde V_n}, t), t) \text{ and } \frac{\partial g(x_{\tilde V_n}, t)}{\partial t} = (\frac{\partial g}{\partial t})(x_{\tilde V_n}(\beta_{\tilde V_n}, t), t)$$

Let us denote

$$\nabla_x g = \nabla_x g(\beta_{\tilde V_n}, t) = (\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_{N+1}}), \quad \dot x = \dot x(\beta_{\tilde V_n}, t) = (\frac{\partial x_1}{\partial t}, \dots, \frac{\partial x_{N+1}}{\partial t})$$

in particular,

$$\nabla_x g . \dot x = \sum_{k=1}^{N+1} \frac{\partial g(x_{\tilde V_n}, t)}{\partial x_k} \frac{\partial x_k(\beta_{\tilde V_n}, t)}{\partial t}.$$

We will show that

$$\int_{\Omega_t} \nabla_x g . \dot x d\beta_{\tilde V_n} = 0.$$

Let

$$\nabla_\beta \Phi_t = \nabla_\beta \Phi_t(\beta_{\tilde V_n}, t) = \left(\frac{\partial x_i}{\partial \beta_j} \right)_{1 \le i,j \le N+1},$$

we have

$$\frac{\frac{d}{dt} \det \left(\nabla_\beta \Phi_t \right)}{\det \nabla_\beta \Phi_t} = \frac{1}{\det \nabla_\beta \Phi_t} \sum_{k=1}^{N+1} \left(\det \left(\frac{\partial x_i}{\partial \beta_j} \mathbb{1}_{i \ne k} + \frac{\partial \dot x_i}{\partial \beta_j} \mathbb{1}_{i=k} \right)_{1 \le i,j \le N+1} \right)$$

For each $1 \le k \le N+1$, developing the matrix $\left(\frac{\partial x_i}{\partial \beta_j} \mathbb{1}_{i \ne k} + \frac{\partial \dot x_i}{\partial \beta_j} \mathbb{1}_{i=k} \right)_{1 \le i,j \le N+1}$ w.r.t. its k-th row, using Cramer's rule to inverse $\nabla_\beta \Phi_t$ with cofactor, we have

$$\frac{\det \left(\frac{\partial x_i}{\partial \beta_j} \mathbb{1}_{i \ne k} + \frac{\partial \dot x_i}{\partial \beta_j} \mathbb{1}_{i=k} \right)_{1 \le i,j \le N+1}}{\det \nabla_\beta \Phi_t} = \sum_{j=1}^{N+1} \frac{\partial \dot x_k}{\partial \beta_j}(\beta_{\tilde V_n}, t) \frac{\partial \beta_j}{\partial x_k}(x_{\tilde V_n}(\beta_{\tilde V_n}, t), t) = \frac{\partial \dot x_k}{\partial x_k}(x_{\tilde V_n}(\beta_{\tilde V_n}, t), t)$$

where we consider $\dot x_k = \dot x_k(\beta_{\tilde V_n}(x_{\tilde V_n}, t), t) = \frac{\partial x_k}{\partial t}(\beta_{\tilde V_n}(x_{\tilde V_n}, t), t)$, it then follows that

$$\frac{\frac{d}{dt} \det \left(\nabla_\beta \Phi_t \right)}{\det \nabla_\beta \Phi_t} = \sum_{k=1}^{N+1} \frac{\partial \dot x_k}{\partial x_k}(x_{\tilde V_n}(\beta_{\tilde V_n}, t), t) = \nabla_x \cdot \dot x(x_{\tilde V_n}(\beta_{\tilde V_n}, t), t)$$

In addition, for any t, $\det \nabla_\beta \Phi_t = 2^{N+1}$, hence $\nabla_x \cdot \dot{x} = 0$, then by denoting $\mathscr{D} = \mathbb{R}^{N+1}$

$$\begin{aligned}
\int_{\Omega_t} \nabla_x g.\dot{x}(\beta_{\tilde{V}_n}, t) d\beta_{\tilde{V}_n} &= \int_{\mathscr{D}} \nabla_x g.\dot{x}(x_{\tilde{V}_n}, t) \det(\nabla_x \Phi_t^{-1}) dx_{\tilde{V}_n} \\
&= \int_{\mathscr{D}} \left(\nabla_x g.\dot{x} + g \nabla_x \cdot \dot{x}\right) 2^{-(N+1)} dx_{\tilde{V}_n} \\
&= \int_{\mathscr{D}} \nabla_x \cdot (g\dot{x}) 2^{-(N+1)} dx_{\tilde{V}_n} \\
&= \int_{\partial\mathscr{D}} (g\dot{x}) 2^{-(N+1)} dx_{\tilde{V}_n} = 0
\end{aligned}$$

where we have applied integration by part for each coordinate where $\partial\mathscr{D}$ is the union of sets $\partial\mathscr{D}_j = \{(u_i) \in \mathbb{R}^{N+1}, \text{s.t. } u_j = 0\}$. We used in the law equality the fact that $(g\dot{x})$ vanishes at the boundary (indeed, we have an exponential bound on g, cf (6.10.69), and $\dot{x}$ is a rational function of x). It then follows that

$$\begin{aligned}
\int_{\Omega_t} \frac{d}{dt} f^{(n)}(\beta_{\tilde{V}_n}, t) d\beta_{\tilde{V}_n} &= \int_{\Omega_t} \frac{\partial g(x_{\tilde{V}_n}, t)}{\partial t} d\beta_{\tilde{V}_n} \\
&= \int_{\mathscr{D}} \frac{\partial g(x_{\tilde{V}_n}, t)}{\partial t} \det(\nabla_x \Phi_t^{-1}) dx_{\tilde{V}_n} \\
&= \frac{\partial}{\partial t} \int_{\Omega_t} f^{(n)}(\beta_{\tilde{V}_n}, t) d\beta_{\tilde{V}_n}.
\end{aligned}$$

Now we proceed to the proof of (6.5.40), denote $V_n = \{1, 2, \ldots, |V_n|\}$ and $V_{n+1} \setminus V_n = \{|V_n|+1, |V_n|+2, \ldots, |V_{n+1}| = N\}$ and $\delta_{n+1} = N+1$. We write $H^{(n+1)}_{\beta,t}$ in the following form

$$\begin{pmatrix}
2\beta_1 & & & & & & \\
 & \cdots & & & & -W^t & \\
 & & 2\beta_{|V_n|} & & & & \\
 & & & 2\beta_{|V_n|+1} & & & \\
 & & & & \cdots & & \\
 & & -W^t & & & 2\beta_N & \\
 & & & & & & 2\beta_{\delta_{n+1}}
\end{pmatrix} \tag{6.10.70}$$

Fix the value β_{V_n} on V_n. Write now, $\tilde{\Omega}_t = \{\beta_i, i \in \tilde{V}_{n+1} \setminus V_n; H^{(n+1)}_{\beta,t} > 0\}$. Note that by the diffeomorphism Φ_t, with the choice of ordering made in (6.10.70), the variables $x_1, \cdots, x_{|V_n|}$ only depends on the variables β_{V_n}. We consider them as fixed in the sequel, as β_{V_n} is fixed. We denote by

$$\tilde{\Phi}_t : \beta_{\tilde{V}_{n+1} \setminus V_n} \in \tilde{\Omega}_t \mapsto x_{\tilde{V}_{n+1} \setminus V_n} \in \mathbb{R}_+^{|\tilde{V}_{n+1} \setminus V_n|},$$

the function obtained from Φ_t with β_{V_n}, $x_1, \cdots, x_{|V_n|}$ fixed. By previous remarks and since Φ_t is a diffeomorphism, $\tilde{\Phi}_t$ is a diffeomorphism. We have by (6.10.66), $\det(\nabla_x \tilde{\Phi}_t^{-1}) = \frac{1}{2^{|\tilde{V}_{n+1} \setminus V_n|}}$. We consider now $f^{(n+1)}(\beta_{\tilde{V}_{n+1} \setminus V_n}, t)$ as a function of t and $\beta_{\tilde{V}_{n+1} \setminus V_n}$ (β_{V_n} is implicite in the notation since it is fixed), let (abusively) $g : \mathbb{R}_+^{|\tilde{V}_{n+1} \setminus V_n|} \times \mathbb{R}_+ \to \mathbb{R}$ be defined as $g(x_{\tilde{V}_{n+1} \setminus V_n}, t) = f^{(n+1)}(\tilde{\Phi}_t^{-1}(x_{\tilde{V}_{n+1} \setminus V_n}), t)$. The rihgt hand side of (6.5.40) is

$$\begin{aligned}
&\int_{\tilde{\Omega}_t} \frac{d}{dt} f^{(n+1)}(\beta_{\tilde{V}_{n+1}\setminus V_n}, t) d\beta_{\tilde{V}_{n+1}\setminus V_n} \\
&= \int_{\tilde{\Omega}_t} \frac{d}{dt} g(x_{\tilde{V}_{n+1}\setminus V_n}(\beta_{\tilde{V}_{n+1}\setminus V_n}, t), t) d\beta_{\tilde{V}_{n+1}\setminus V_n} \\
&= \int_{\tilde{\Omega}_t} \left[\sum_{k=|V_n|+1}^{N+1} \frac{\partial g(x_{\tilde{V}_{n+1}\setminus V_n}, t)}{\partial x_k} \frac{\partial x_k(\beta_{\tilde{V}_{n+1}\setminus V_n}, t)}{\partial t} + \frac{\partial g(x_{\tilde{V}_{n+1}\setminus V_n}, t)}{\partial t} \right] d\beta_{\tilde{V}_{n+1}\setminus V_n}.
\end{aligned}$$

As before, the contribution of

$$\int_{\tilde{\Omega}_t} \sum_{k=|V_n|+1}^{N+1} \frac{\partial g}{\partial x_k}(x_{\tilde{V}_{n+1}\setminus V_n}(\beta_{\tilde{V}_{n+1}\setminus V_n}, t), t) \frac{\partial x_k(\beta_{\tilde{V}_{n+1}\setminus V_n}, t)}{\partial t} d\beta_{\tilde{V}_{n+1}\setminus V_n}$$

is zero for the same reasons. Therefore,

$$\begin{aligned}
\int_{\tilde{\Omega}_t} \frac{d}{dt} f^{(n+1)}(\beta_{\tilde{V}_{n+1}\setminus V_n}, t) d\beta_{\tilde{V}_{n+1}\setminus V_n} &= \int_{\tilde{\Omega}_t} \frac{\partial g(x_{\tilde{V}_{n+1}\setminus V_n}, t)}{\partial t} (\beta_{\tilde{V}_{n+1}\setminus V_n}, t) d\beta_{\tilde{V}_{n+1}\setminus V_n} \\
&= \int_{\mathbb{R}_+^{|\tilde{V}_{n+1}\setminus V_n|}} \frac{d}{dt} g(x_{\tilde{V}_{n+1}\setminus V_n}, t) \det(\nabla_x \tilde{\Phi}_t^{-1}) dx_{\tilde{V}_{n+1}\setminus V_n} \\
&= \frac{d}{dt} \int_{\mathbb{R}_+^{|\tilde{V}_{n+1}\setminus V_n|}} g(x_{\tilde{V}_{n+1}\setminus V_n}, t) \det(\nabla_x \tilde{\Phi}_t^{-1}) dx_{\tilde{V}_{n+1}\setminus V_n} \\
&= \frac{d}{dt} \int_{\Omega_t^{\tilde{V}_n\setminus V_n}} f^{(n+1)}(\beta_{\tilde{V}_{n+1}\setminus V_n}, t) d\beta_{\tilde{V}_{n+1}\setminus V_n}.
\end{aligned}$$

where the application of dominated convergence theorem is argued in the same way of (6.10.68), since the expression of $\beta_i, i \in \tilde{V}_{n+1} \setminus V_n$ in terms of $x_{\tilde{V}_{n+1}\setminus V_n}$ are exactly the same as in (6.10.66).

□

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Marches aléatoires renforcées et opérateurs de Schrödinger aléatoires

Résumé: Cette thèse s'intéresse à deux modèles de processus auto intéagissant étroitement reliés: le processus de sauts renforcé par sites (VRJP) et la marche aléatoire renforcée par arêtes (ERRW). Nous étudions aussi les liens entre ces processus et un opérateur de Schrödinger aléatoire.
Dans le chapitre 3, nous montrons que le VRJP est le seul processus satisfaisant la propriété d'échangeabilité partielle et tel que la probabilité de transition ne dépende que du temps local des voisins, sous quelques conditions techniques.
Le chapitre 4 donne la transition de phase entre vitesse positive et vitesse nulle pour un VRJP transitoire sur un arbre de Galton Watson, utilisant le fait que sur un arbre, le VRJP est une marche aléatoire en milieu aléatoire.
Dans le chapitre 5, une nouvelle famille exponentielle de loi est introduite et ses liens avec le VRJP sont étudiés. En particulier, nous donnons une preuve de la formule de Coppersmith et Diaconis, n'utilisant que des calculs élémentaires.
Finalement, dans le chapitre 6 nous étudions la représentation du VRJP comme mélange de processus de Markov sur les graphes infinis. Nous représentons le VRJP à l'aide de la fonction de Green et d'une fonction propre généralisée d'un opérateur de Schrödinger aléatoire associé au VRJP. En conséquence, nous obtenons un principe d'invariance pour le VRJP quand le renforcement est suffisamment faible, ainsi que la récurrence du ERRW sur $\mathbb{Z}^2$ pour toute valeurs initiales des paramètres.

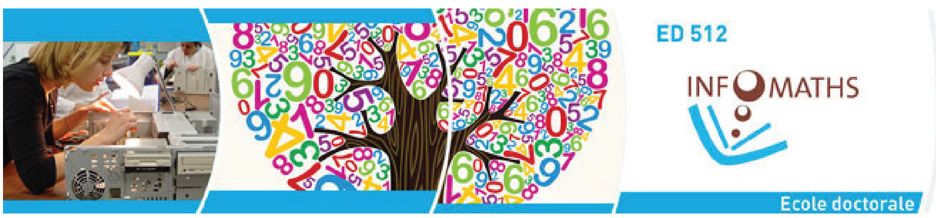
Mots clés: Marches aléatoires renforcées, marches aléatoires en milieux aléatoires, Schrödinger aléatoires.

Reinforced random walks and Random Schrödinger operators

Abstract: This thesis is dedicated to the study of two closely related self-interacting processes: the vertex reinforced jump process (VRJP) and the edge reinforced random walk (ERRW). We also study the relations between these processes and a random Schrödinger operator.
In Chapter 3, we prove that the VRJP is the only partially exchangeable process whose transition probability depends only on neighbor local times, under some technical conditions.
Chapter 4 gives the phase transition between positive speed and null speed of a transient VRJP on a Galton Watson tree, using a representation of random walk in independent random environment.
In Chapter 5, we introduce a new exponential family of probability distributions generalizing the Inverse Gaussian distribution, and we show some of its relations to the VRJP. In particular, we give an elementary proof of the formula of Coppersmith and Diaconis.
Finally, we show in Chapter 6 that the VRJP on infinite graph is a mixture of Markov jump processes, by constructing the random environment using the Green function and a generalized eigenfunction related to a random Schrödinger operator associated with the VRJP. As a consequence, we obtain a central limit theorem when the reinforcement is weak enough, and also the recurrence of ERRW on $\mathbb{Z}^2$ for any initial constant weights.

Keywords: Reinforced random walks, random walks in random environments, random Schrödinger.

Image en couverture : Première rencontre avec un magicien et sa formule magique.



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